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Worst VaR scenarios with given marginals and measures of association

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ABSTRACT

This paper studies the problem of finding best-possible upper bounds on the Value-at-Risk for a function of two random variables when the marginal distributions are known and additional nonparametric information on the dependence structure, such as the value of a measure of association, is available. The same problem for the Tail-Value-at-Risk is also briefly discussed.

1. Introduction

This paper studies generalized versions of the following problem, which was attributed to A.N. Kolmogorov by Makarov (1981): Let X and Y be two random variables with given distribution functions F_1 and F_2, respectively. Let G_ψ denote the distribution function of a function $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ (e.g., the sum) of X and Y. Find $\underline{G}_\psi(x) = \inf\{G_\psi(x)\}$, where the infimum is taken over the *Fréchet–Hoeffding class* $\mathbf{F}(F_1, F_2)$ consisting of all joint distribution functions with marginals F_1 and F_2. In this paper, we will be concerned with generalized versions allowing for partial information on the dependence structure between X and Y, in which case the infimum is taken over well-defined subclasses of $\mathbf{F}(F_1, F_2)$.

Besides being theoretically challenging, the problem described above is highly relevant in risk management: it is equivalent to finding worst case scenarios for the Value-at-Risk for a function of two risks (random variables) when the marginal distributions of the risks are known but the dependence structure between the risks is unknown, or, as in this paper, only partially known.

In the complete absence of partial information on the dependence structure, the problem of Kolmogorov was solved by Makarov (1981) for the case that G is the distribution function of the sum of X and Y with given marginals F_1 and F_2. Using different routes, the same problem was solved by Rüschendorf (1982, dual approach) and by Frank et al. (1987, copula-based approach).

The presence of partial information on the dependence structure may conceivably allow obtaining tighter bounds. For instance, if a lower bound on the copula of X and Y – sharper than the Fréchet–Hoeffding lower bound – is available, Williamson and Downs (1990) provide a general method to produce tighter bounds; see Section 2 for details. Such a lower bound need not be described parametrically or even have a closed-form expression. It could, for example, be represented in a lookup table. It motivates the search for lower bounds on the copula. However, in many

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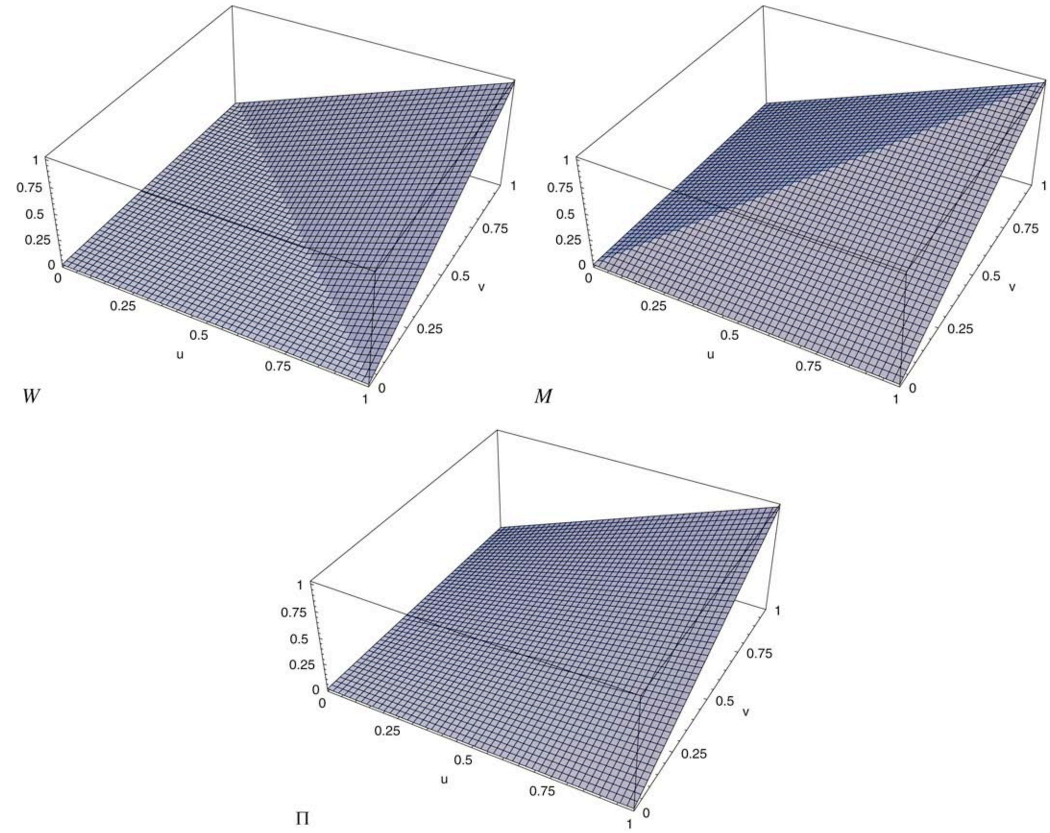

Fig. 1. The Fréchet–Hoeffding bounds and the product copula.

cases, partial information on the dependence structure does not trivially translate into a lower bound on the copula. In this paper we will carry out some of such non-trivial translations, establishing some new copula bounds in the presence of partial nonparametric dependence information.

One of our aims is to investigate and illustrate the effectiveness (or in some cases rather, lack thereof) of different types of information on the dependence structure in bounding the Value-at-Risk. To some extent the paper is meant as an educative warning that, contrary to what is quite often thought in the insurance and financial industry, the Value-at-Risk may vary widely even when the marginals and a nonparametric dependence measure, such as the value of a measure of association, are fixed. Readers should thus be careful of adopting in this context the commonly used multivariate inference techniques that *implicitly* assume otherwise. The same problem for the Tail-Value-at-Risk is also briefly discussed.

Other work related to the problem of Kolmogorov includes, without being exhaustive, Denuit et al. (1999), Nelsen et al. (2001, 2004), Dhaene et al. (2002), Nelsen and Úbeda Flores (2004), Denuit et al. (2005), Embrechts et al. (2005), Embrechts and Puccetti (2006) and Laeven (2009).

The outline of this paper is as follows: In Section 2 we briefly review some probabilistic theory of copulas in view of the problem under study. In Sections 3–5 we derive new copula bounds in the presence of nonparametric information on the dependence structure. In Section 6 we use these copula bounds to obtain bounds on the Value-at-Risk and study the effectiveness of different types of nonparametric dependence information in bounding the Value-at-Risk. Section 7 briefly studies worst Tail-Value-at-Risk scenarios. We finish with an open problem in Section 8 and some concluding remarks in Section 9.

2. Preliminaries

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider two continuous random variables (r.v.'s) X and Y defined on it. We denote by $F(x, y) = \mathbb{P}[X \leq x, Y \leq y]$ the joint distribution function (d.f.) of (X, Y) and denote by $F_1(x) = \mathbb{P}[X \leq x]$ and $F_2(y) = \mathbb{P}[Y \leq y]$ the marginal d.f.'s. We assume throughout that the marginal d.f.'s are given while the joint d.f. is unknown.

A bivariate copula is a function $C : [0, 1]^2 \to [0, 1]$ that satisfies the boundary conditions

$$C(u, 0) = C(0, v) = 0,$$
$$C(u, 1) = u, \qquad C(1, v) = v,$$

for every $u, v \in [0, 1]$, and is 2-increasing on $[0, 1]^2$, i.e.,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0, \tag{1}$$

for all $u_1, u_2, v_1, v_2 \in [0, 1]$ with $u_1 \leq u_2$ and $v_1 \leq v_2$. Equivalently, a bivariate copula is a bivariate d.f. with domain $[0, 1]^2$ and uniform $(0, 1)$ marginals. Every bivariate copula C satisfies the *Fréchet–Hoeffding inequality*

$$\begin{aligned} W(u, v) &= \max\{0;\ u + v - 1\} \\ &\leq C(u, v) \leq \min\{u; v\} = M(u, v), \end{aligned}$$

for every $(u, v) \in [0, 1]^2$. Here, the Fréchet–Hoeffding bounds W and M are themselves bivariate copulas. A third copula that plays an important role is the *product copula* $\Pi(u, v) = uv$. Fig. 1 plots the copulas W, M and Π.

It is straightforward to verify that for a given bivariate copula C and given marginals F_1 and F_2, the function F defined by

$$F(x, y) = C(F_1(x), F_2(y)), \quad (x, y) \in \mathbb{R}^2, \tag{2}$$

is a bivariate d.f. with marginals F_1 and F_2. Sklar (1959) proved that also the converse is true: for a given bivariate d.f. F with marginals F_1 and F_2 there exists a bivariate copula C satisfying (2). Whenever F_1 and F_2 are continuous, C is unique. Because only bivariate problems are studied in this paper we will henceforth refer to a bivariate copula as a copula.

For later reference, we also define the *dual of a copula*: the dual of a copula C is the function $\widetilde{C}$ defined by $\widetilde{C}(u,v) = u+v-C(u,v)$. We note that $\widetilde{C}$ is not a copula. However, when C is the copula of a pair of r.v.'s X and Y (in the sense of Sklar),

$$\mathbb{P}[X \le x \text{ or } Y \le y] = \widetilde{C}(F_1(x), F_2(y)).$$

In what follows, we consider three measures of association: Kendall's tau, Spearman's rho and Blomqvist's beta. Let (X_1, Y_1) and (X_2, Y_2) be two independent and identically distributed random vectors, with joint d.f. F. Then the population version of Kendall's tau is defined by

$$\tau = \tau_{X,Y} = \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) < 0];$$

it is the probability of *concordance* minus the probability of *discordance*. The population version of Spearman's rho is simply the linear (or Pearson's product-moment) correlation coefficient of $F_1(X)$ and $F_2(Y)$. Denote by (X_3, Y_3) another random vector with joint d.f. F that is independent of (X_1, Y_1) and (X_2, Y_2), then an elegant formulation of Spearman's rho in terms of concordance and discordance probabilities is

$$\rho = \rho_{X,Y} = 3\left(\mathbb{P}[(X_1 - X_2)(Y_1 - Y_3) > 0] - \mathbb{P}[(X_1 - X_2)(Y_1 - Y_3) < 0]\right);$$

the pair (X_3, Y_2) could be used equally as well. Finally, Blomqvist's beta is defined by

$$\beta = \beta_{X,Y} = \mathbb{P}[(X - \tilde{x})(Y - \tilde{y}) > 0] - \mathbb{P}[(X - \tilde{x})(Y - \tilde{y}) < 0],$$

where $\tilde{x}$ and $\tilde{y}$ denote medians of X and Y, respectively.

When X and Y are continuous r.v.'s with copula C, Kendall's tau, Spearman's rho and Blomqvist's beta for X and Y are functions of C, and are given by

$$\tau_{X,Y} = \tau(C) = 4\int\int_{[0,1]^2} C(u,v)\mathrm{d}C(u,v) - 1; \tag{3}$$

$$\rho_{X,Y} = \rho(C) = 12\int\int_{[0,1]^2} C(u,v)\mathrm{d}u\mathrm{d}v - 3; \tag{4}$$

$$\beta_{X,Y} = \beta(C) = 4C(1/2, 1/2) - 1. \tag{5}$$

Knowing the value of a measure of association may allow one to improve the Fréchet–Hoeffding bounds. Nelsen et al. (2001) and Nelsen and Úbeda Flores (2004) derive pointwise best-possible lower bounds on the bivariate copula for given values of Kendall's tau, Spearman's rho or Blomqvist's beta. In particular, they show that if $\underline{C}_\alpha(u,v) = \inf\{C(u,v) \mid C \text{ is a copula}, \ \alpha(C) = \alpha\}$,

$$\underline{C}_\tau(u,v) = \max\left\{0;\ u+v-1;\ \frac{1}{2}\left(u+v-\sqrt{(u-v)^2+1-\tau}\right)\right\}; \tag{6}$$

$$\underline{C}_\rho(u,v) = \max\left\{0;\ u+v-1;\ \frac{1}{2}\left(u+v-\phi(u,v,\rho)\right)\right\}; \tag{7}$$

$$\underline{C}_\beta(u,v) = \max\left\{0;\ u+v-1;\ \frac{\beta+1}{4} - \left(\frac{1}{2}-u\right)_+ - \left(\frac{1}{2}-v\right)_+\right\}; \tag{8}$$

with

$$\phi(u,v,\rho) = \frac{1}{3}\left(\left(9(1-\rho) + 3\sqrt{9(1-\rho)^2 - 3(u-v)^6}\right)^{\frac{1}{3}} + \left(9(1-\rho) - 3\sqrt{9(1-\rho)^2 - 3(u-v)^6}\right)^{\frac{1}{3}}\right),$$

and $x_+ = \max\{0;\ x\}$. Notice that whenever the value of Kendall's tau is positive, $\underline{C}_\tau$ is an improvement over W. Furthermore, whenever Spearman's rho is larger than $-\frac{1}{2}$, $\underline{C}_\rho$ is an improvement over W. Finally, for any $\beta > -1$, $\underline{C}_\beta$ is an improvement over W.

The copula lower bounds $\underline{C}_\alpha$, $\alpha \in \{\tau, \rho, \beta\}$, are pointwise best-possible (or sharp) meaning that they are attained and therefore cannot be made narrower: for any given $(u,v) \in [0,1]^2$ and any given value α for the measure of association under consideration, there exists a copula $C_{u,v,\alpha}$ satisfying $\alpha(C_{u,v,\alpha}) = \alpha$ such that $C_{u,v,\alpha}(u,v) = \underline{C}_\alpha(u,v)$ and $C_{u,v,\alpha}(u',v') \ge \underline{C}_\alpha(u',v')$ for all $(u',v') \in [0,1]^2$.

Remark 2.1 (*On $\underline{C}_\rho$ and Complex Values*). If one lets $a = u - v$ and $b = 1 - \rho$, then ϕ has the form

$$3\phi(a,b) = \left(9b + 3\sqrt{9b^2 - 3a^6}\right)^{\frac{1}{3}} + \left(9b - 3\sqrt{9b^2 - 3a^6}\right)^{\frac{1}{3}}.$$

When $9b^2 - 3a^6 \ge 0$, ϕ is clearly real-valued. When $9b^2 - 3a^6 = -d^2 < 0$ (with $d > 0$), then 3ϕ can be written as $3\phi(a,b) = z^{1/3} + \bar{z}^{1/3}$, where $z = 9b + 3id$. Because $\arg z$ and $\arg \bar{z}$ both lie in $(-\pi, \pi)$ and $\arg \bar{z} = -\arg z$, using the principal value of the cube root, $z^{1/3}$ and $\bar{z}^{1/3}$ are themselves complex conjugates. Hence 3ϕ is real-valued. Therefore, with $(\cdot)^{1/3}$ denoting the principal value of the cube root when the argument is not real-valued, $\underline{C}_\rho$ is always real-valued.

It will be useful (in the proof of Theorem 3.2) to know that ϕ is non-negative. When $9b^2 - 3a^6 \ge 0$, $9b^2 - 3a^6$ lies between 0 and $3b$, so that $\phi(a,b) \ge 0$. In the complex case, the real parts of z and $\bar{z}$ are both positive since $\arg z^{1/3}$ and $\arg \bar{z}^{1/3}$ both lie in $(-\pi/3, \pi/3)$.

We note that $\underline{C}_\tau$ and $\underline{C}_\rho$ are copulas, but $\tau(\underline{C}_\tau) < \tau$ and $\rho(\underline{C}_\rho) < \rho$ for all $\tau, \rho \in (-1, 1)$. On the contrary, the copula $\underline{C}_\beta$ satisfies $\beta(\underline{C}_\beta) = \beta$. Figs. 2 and 3 plot the copulas $\underline{C}_\tau$, $\underline{C}_\rho$ and $\underline{C}_\beta$ for given values of τ, ρ and β.

When a lower bound on the copula is available, following Williamson and Downs (1990), pointwise best-possible upper and lower bounds on a function of the r.v.'s X and Y can be derived. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function that is non-decreasing in both coordinates and left-continuous in the second coordinate. Furthermore, let the copula $\underline{C}$ be such that

$$F(x,y) \ge \underline{C}(F_1(x), F_2(y)), \tag{9}$$

for all $(x,y) \in \mathbb{R}^2$. We denote by F_i^- the left-continuous version of the d.f. F_i, $i = 1, 2$. From Embrechts et al. (2003, Theorems 3.1 and 3.2) and Embrechts and Puccetti (2006, Theorem 3.1), we have that for any given number $s \in \mathbb{R}$

$$\sup_{x\in\mathbb{R}}\left\{\underline{C}(F_1(x), F_2^-(\psi_{\hat{x}}(s)))\right\} \le \mathbb{P}[\psi(X,Y) < s] \tag{10}$$

$$\le \mathbb{P}[\psi(X,Y) \le s] \le \inf_{x\in\mathbb{R}}\left\{\widetilde{\underline{C}}(F_1(x), F_2(\psi_{\tilde{x}}(s)))\right\}. \tag{11}$$

Here

$$\psi_{\hat{x}}(s) = \sup\{y \in \mathbb{R} \mid \psi(x,y) < s\}, \quad \text{and}$$
$$\psi_{\tilde{x}}(s) = \sup\{y \in \mathbb{R} \mid \psi(x,y) \le s\},$$

for fixed $x \in \mathbb{R}$. Moreover, there always exist copulas $C_{\underline{t}}$ and $C_{\bar{t}}$ (dependent only on $\underline{t}$ and $\bar{t}$, respectively) such that if the joint d.f.

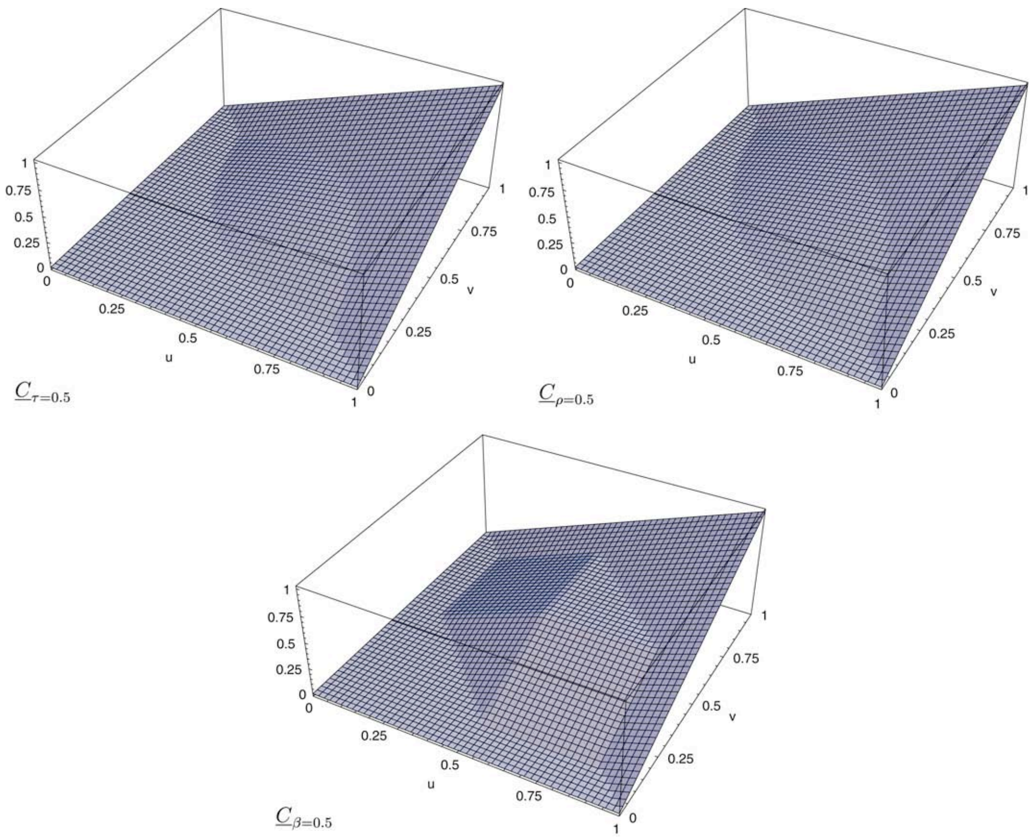

Fig. 2. Lower bounds on the copula for a given τ, ρ or β.

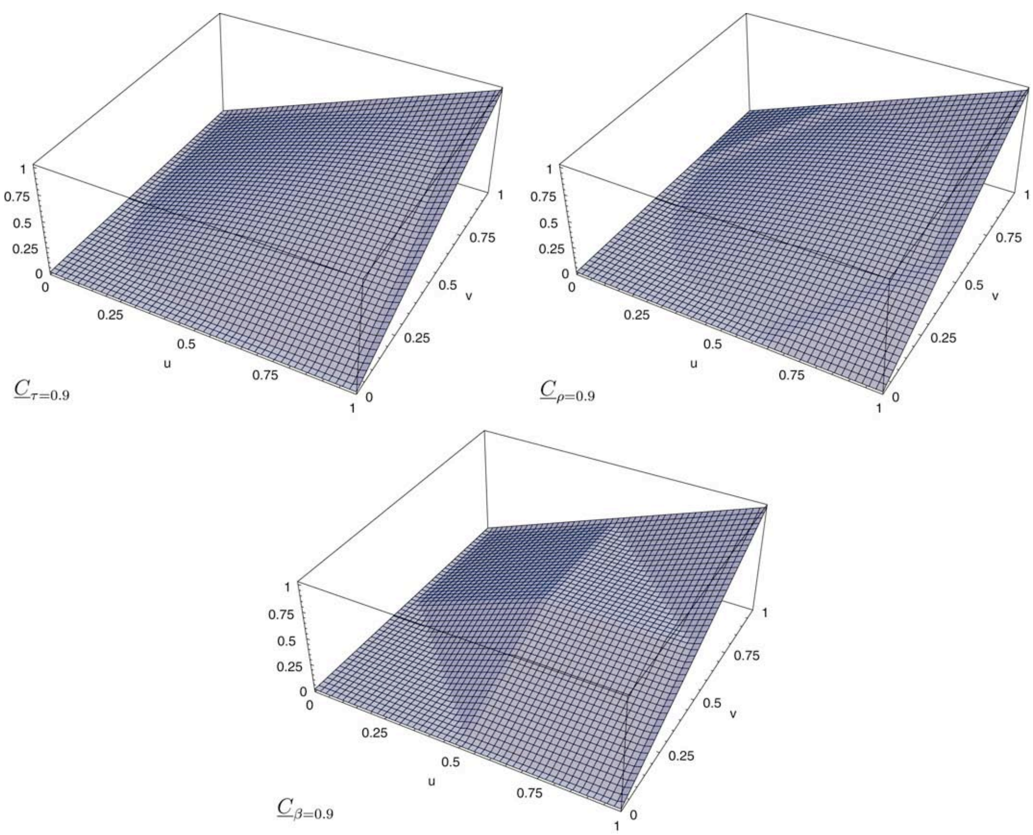

Fig. 3. Lower bounds on the copula for a given τ, ρ or β.

of X and Y is $C_{\underline{t}}(F_1(x), F_2(y))$ then

$$\sup_{x\in\mathbb{R}} \left\{\underline{C}(F_1(x), F_2^-(\psi_{\widehat{x}}(s)))\right\} = \mathbb{P}[\psi(X, Y) < s] = t,$$

while if the joint d.f. is $C_{\widetilde{t}}(F_1(x), F_2(y))$ then

$$\inf_{x\in\mathbb{R}} \left\{\underline{C}(F_1(x), F_2(\psi_{\widehat{x}}(s)))\right\} = \mathbb{P}[\psi(X, Y) \le s] = t.$$

Example 2.1. Let $\psi(x, y) = x+y$ and let F_1 and F_2 be uniform $(0, 1)$ laws. Furthermore, let $\underline{C}$ be as in (6) for a given number τ. In this case, $\psi_{\widehat{x}}(s) = s - x$ and the LHS of (10) becomes

$$\sup_{u\in(0,1)} \left\{\max\left\{0;\ s-1;\ \frac{1}{2}\left(s - \sqrt{(2u-s)^2 + 1 - \tau}\right)\right\}\right\}.$$

Hence, we find that

$$\inf\left\{\mathbb{P}[U + V < s] \mid \tau_{U,V}\right\} = \max\left\{0;\ s-1;\ \frac{1}{2}\left(s - \sqrt{1-\tau_{U,V}}\right)\right\},$$

with U and V two uniform $(0, 1)$ r.v.'s having a Kendall's tau of $\tau_{U,V}$.

We state the following definition:

Definition 2.1. The Value-at-Risk (VaR) at probability level $p \in [0, 1]$ for a r.v. Z with d.f. G is defined as

$$\mathrm{VaR}_p[Z] = \inf\left\{x \in \mathbb{R} \mid G(x) \ge p\right\}, \tag{12}$$

where, as usual, $\inf\{\emptyset\} = +\infty$ by convention. Notice that VaR_p is a left-continuous function of p. It is the generalized inverse of G, meaning that upon inversion bounds on G transform into equivalent bounds on VaR_p.

Remark 2.2. Notice that the lower bound $\underline{C}$ determines both the lower and (via its dual $\widetilde{\underline{C}}$) the upper bound on the probabilities in (10) and (11). Consequently, upper and lower bounds on the VaR are both determined by a lower bound on the copula.

Remark 2.3. Whenever $\underline{C}$ satisfies (9) but is not a copula, the bounds in (10) and (11), hence the equivalent bounds on $\mathrm{VaR}_p[\psi(X, Y)]$, are bounds but may fail to be sharp.

3. When the risks are PQD and τ or ρ is given

The r.v.'s X and Y are said to be *positively quadrant dependent* (PQD) if for all $(x, y) \in \mathbb{R}^2$

$$\mathbb{P}[X \le x, Y \le y] \ge \mathbb{P}[X \le x]\mathbb{P}[Y \le y],$$

or equivalently,

$$\mathbb{P}[Y \le y \mid X \le x] \ge \mathbb{P}[Y \le y].$$

When X and Y are continuous r.v.'s with copula C they are PQD if and only if

$$C(u, v) \ge \Pi(u, v),$$

for all $(u, v) \in [0, 1]^2$. As is well known, $C \ge \Pi$ implies that $\tau(C), \rho(C) \ge 0$. However, the converse is not true.

Some remarks:

Remark 3.1. We warn the reader that $C \ge \Pi$ is a rather strong assumption and assuming it while "not fully" appropriate may lead to a significant undervaluation of the worst VaR scenario. This is a consequence of the fact that the notion of PQD establishes a partial rather than complete order in the class of bivariate copulas. Assuming $C \ge \Pi$ excludes all copulas that are not comparable to Π.

Remark 3.2. The notion of PQD is implied by many other positive dependence notions such as tail monotonicity, stochastic monotonicity, corner set monotonicity and likelihood ratio dependence; see Section 5.2 of Nelsen (1999) for details.

Remark 3.3. The interested reader is referred to Denuit and Scaillet (2004) for nonparametric tests for PQD.

Remark 3.4. As is well known (see e.g., Proposition 5.3.9 of Denuit et al. (2005) or Dhaene and Goovaerts (1996); and Proposition 3.4.29 of Denuit et al. (2005) or Wang and Dhaene (1998)), when the r.v.'s X and Y are PQD,

$$\mathbb{E}[(X^\perp + Y^\perp - d)_+] \le \mathbb{E}[(X + Y - d)_+] \le \mathbb{E}[(X^c + Y^c - d)_+],$$

for all $d \in \mathbb{R}$ provided that the expectations exist. Here, the random vectors $(X^\perp, Y^\perp)$ and (X^c, Y^c) have the same marginal d.f.'s as the random vector (X, Y), $X^\perp$ and $Y^\perp$ are independent r.v.'s, and X^c and Y^c are comonotonic r.v.'s. In actuarial mathematics, $\mathbb{E}[(X+Y-d)_+]$ is known as the stop-loss premium of $X + Y$ at a retention level d.

We state the following theorem:

Theorem 3.1. *Let C be a copula and suppose that $\tau(C) = \tau \ge 0$ and that $C \ge \Pi$. Then $C(u, v) \ge \underline{C}_{\mathrm{PQD},\tau}(u, v)$, for all $(u, v) \in [0, 1]^2$, with*

$$\underline{C}_{\mathrm{PQD},\tau}(u, v) = \max\left\{uv;\ \frac{1}{2}\left(u + v - \sqrt{(u-v)^2 + 1 - \tau}\right)\right\}. \tag{13}$$

The bound $\underline{C}_{\mathrm{PQD},\tau}$ is pointwise best-possible and is itself a copula.

Proof. Fix $\tau \in [0, 1]$. Theorem 3.2.2 of Nelsen (1999) states that any copula C satisfying $C(a, b) = \theta$ for given values $(a, b) \in [0, 1]^2$ and $\theta \in [0, 1]$ must satisfy $C(u, v) \le \overline{C}_{(a,b),\theta}(u, v)$, for all $(u, v) \in [0, 1]^2$, with

$$\overline{C}_{(a,b),\theta}(u, v) = \begin{cases} \min\{u;\ v;\ \theta\}, & [0, a] \times [0, b]; \\ \min\{u;\ v - b + \theta\}, & [0, a] \times [b, 1]; \\ \min\{v;\ u - a + \theta\}, & [a, 1] \times [0, b]; \\ \min\{u;\ v;\ u - a + v - b + \theta\}, & [a, 1] \times [b, 1]. \end{cases}$$

One easily verifies that $\overline{C}_{(a,b),\theta} \ge \Pi$ (assuming $\theta \ge ab$) and therefore the additional information that $C \ge \Pi$ would not improve this upper bound.

Following the proof of Theorem 2 of Nelsen et al. (2001), we calculate Kendall's tau for $\overline{C}_{(a,b),\theta}$: $\tau(\overline{C}_{(a,b),\theta}) = 1 - 4(a - \theta)(b - \theta)$. Notice that $C \le \overline{C}_{(a,b),\theta}$ implies that $\tau(C) \le \tau(\overline{C}_{(a,b),\theta})$ or equivalently $\tau(C) \le 1 - 4(a - \theta)(b - \theta)$. Solving for the feasible root of $1 - 4(a - \lambda)(b - \lambda) = \tau$ yields $\lambda = \frac{1}{2}\left(a + b - \sqrt{(a-b)^2 + 1 - \tau}\right)$. It follows that $\theta \ge \lambda$ and consequently that $C(a, b) \ge \max\{ab;\ \lambda\}$.

To prove that (13) is pointwise best-possible we need to prove that for any $(a, b) \in [0, 1]^2$ there exists a copula C satisfying $\tau(C) = \tau$ and $C \ge \Pi$ such that $C(a, b) = \max\{ab;\ \lambda\}$. We distinguish between two cases. Case 1: $\lambda \ge ab$. For this case $\overline{C}_{(a,b),\lambda}$ satisfies $\tau(\overline{C}_{(a,b),\lambda}) = \tau$, $\overline{C}_{(a,b),\lambda} \ge \Pi$ and $\overline{C}_{(a,b),\lambda}(a, b) = \lambda$. Case 2: $\lambda < ab$. For this case, $C_\alpha = \alpha\overline{C}_{(a,b),ab} + (1 - \alpha)\Pi$, $\alpha \in [0, 1]$, is a family of copulas of which each member satisfies $C_\alpha \ge \Pi$ and $C_\alpha(a, b) = ab$. Furthermore, $\tau(C_\alpha)$ is a continuous function of α satisfying $0 = \tau(\Pi) \le \tau(C_\alpha) \le \tau(\overline{C}_{(a,b),ab})$. It now follows from the intermediate value theorem that there exists an $\alpha \in [0, 1]$ such that $\tau(C_\alpha) = \tau$.

We finally prove that $\underline{C}_{\mathrm{PQD},\tau}$ is a copula. We will show directly that the 2-increasing condition is satisfied, i.e., the $\underline{C}_{\mathrm{PQD},\tau}$-volume

V of every rectangle $[a, b] \times [c, d]$ is non-negative. Let us write $\underline{C}_{\mathrm{PQD},\tau}(u, v) = \max\{uv;\ f(u, v)\}$ where

$$f(u, v) = \frac{1}{2}\left(u + v - \sqrt{(u - v)^2 + 1 - \tau}\right).$$

The boundary between the subset of $[0, 1]^2$ where $\underline{C}_{\mathrm{PQD},\tau}(u, v) = uv$ and where $\underline{C}_{\mathrm{PQD},\tau}(u, v) = f(u, v)$ is the curve $uv = f(u, v)$, or $4uv(1-u)(1-v) = 1-\tau$, with $\underline{C}_{\mathrm{PQD},\tau}(u, v) = f(u, v)$ inside the curve and $\underline{C}_{\mathrm{PQD},\tau}(u, v) = uv$ outside. If $[a, b] \times [c, d]$ lies entirely outside the curve, then clearly $V([a, b]\times[c, d]) \ge 0$. If $[a, b]\times[c, d]$ lies entirely inside the curve, then $V([a, b] \times [c, d]) \ge 0$ since $\partial^2 f/\partial u\partial v = \frac{1}{2}\left((1-\tau)/[(u-v)^2 + 1 - \tau]^{3/2}\right) \ge 0$. The only other rectangles that need to be considered are those where the boundary curve enters the rectangle at one corner and exits the rectangle at the diagonally opposite corner. For example, if the curve enters the rectangle $[a, b] \times [c, d]$ at (a, d) and exits at (b, c) and $\underline{C}_{\mathrm{PQD},\tau}(u, v) = uv$ below the curve and $\underline{C}_{\mathrm{PQD},\tau}(u, v) = f(u, v)$ above the curve, then

$$\begin{aligned} V([a, b] \times [c, d]) &= f(b, d) - ad - bc + ac \\ &\ge bd - ad - ab + ac \\ &= (b-a)(d-c) \ge 0. \end{aligned}$$

Other cases are similar. This completes the proof. □

We note that $\underline{C}_{\mathrm{PQD},\tau}$ is an improvement over Π only for large values of τ ($\tau > \frac{3}{4}$).

Theorem 3.2. *Let C be a copula and suppose that $\rho(C) = \rho \ge 0$ and that $C \ge \Pi$. Then $C(u, v) \ge \underline{C}_{\mathrm{PQD},\rho}(u, v)$, for all $(u, v) \in [0, 1]^2$, with*

$$\underline{C}_{\mathrm{PQD},\rho}(u, v) = \max\left\{uv;\ \frac{1}{2}\left(u + v - \phi(u, v, \rho)\right)\right\}. \tag{14}$$

The bound $\underline{C}_{\mathrm{PQD},\rho}$ is pointwise best-possible and is itself a copula.

Proof. The first part of the proof of this theorem mimics the proof of Theorem 3.1. To prove that $\underline{C}_{\mathrm{PQD},\rho}$ is a copula let us write $\underline{C}_{\mathrm{PQD},\rho}(u, v) = \max\{uv;\ g(u, v)\}$ where $g(u, v) = \frac{1}{2}(u + v - \phi(u, v))$. Here the boundary curve $uv = g(u, v)$ is $6uv(1-u)(1-v)(u+v-2uv) = 1-\rho$. To complete the proof we need to show that (see also the proof of Theorem 3.1) $\partial^2 g/\partial u\partial v \ge 0$ (or equivalently, $\partial^2\phi/\partial u\partial v \le 0$) inside the curve. But since ϕ satisfies the equation $\phi^3 - (u-v)^2\phi = 2(1-\rho)/3$, we have

$$\frac{\partial^2\phi}{\partial u\partial v} = \frac{-6\phi[3\phi^2 + (u-v)^2][\phi^2 - (u-v)^2]}{[3\phi^2 - (u-v)^2]}.$$

When $\phi = 0$, $\partial^2\phi/\partial u\partial v = 0$. When $\phi > 0$, $\phi^2 - (u-v)^2 = 2(1-\rho)/3\phi > 0$, so that $3\phi^2 - (u-v)^2 > 0$ as well. Thus $\partial^2\phi/\partial u\partial v \le 0$. □

We note that $\underline{C}_{\mathrm{PQD},\rho}$ is an improvement over Π only for large values of ρ ($\rho > \frac{13}{16}$). Fig. 4 plots the copulas $\underline{C}_{\mathrm{PQD},\tau}$ and $\underline{C}_{\mathrm{PQD},\rho}$ for given values of τ and ρ.

4. When both τ or ρ and β are given

We state the following theorem:

Theorem 4.1. *Let C be a copula and suppose that $\tau(C) = \tau$ and that $C(\frac{1}{2}, \frac{1}{2}) = \frac{\beta+1}{4} = \xi$. Then $C(u, v) \ge \underline{C}_{\tau,\beta}(u, v)$, for all $(u, v) \in [\frac{1}{2}, 1]^2$, where*

$$\underline{C}_{\tau,\beta}(u, v) = \max\left\{u + v - 1;\ \xi;\ \frac{1}{2}\left(u + v - \sqrt{(u-v)^2 - \tau + 4\left(\xi - \xi^2\right)}\right)\right\}. \tag{15}$$

The bound $\underline{C}_{\tau,\beta}$ is pointwise best-possible whenever $\underline{C}_{\tau,\beta}(u, v) + \xi \ge 1$.

Proof. Suppose that a copula C satisfies $C(\frac{1}{2}, \frac{1}{2}) = \frac{\beta+1}{4} = \xi$ and $C(a, b) = \theta$ for given values $\beta \in [0, 1]$, $(a, b) \in [0, 1]^2$ and $\theta \in [0, 1]$. We will first derive an upper bound on C, extending Theorem 3.2.2 of Nelsen (1999). We assume $a, b > \frac{1}{2}$; the other three possible domains are similar. We distinguish between the following cases:

1. $(u, v) \in [0, \frac{1}{2}]\times[0, \frac{1}{2}]$: Then $C(u, v) \le C(\frac{1}{2}, \frac{1}{2}) = \xi$, and hence $C(u, v) \le \min\{u;\ v;\ \xi\}$.
2. $(u, v) \in [\frac{1}{2}, 1]\times[0, \frac{1}{2}]$: Then $C(u, v) \le C(u, \frac{1}{2}) \le \xi + u - \frac{1}{2} \le u$, and hence $C(u, v) \le \min\left\{v;\ \xi + u - \frac{1}{2}\right\}$.
3. $(u, v) \in [0, \frac{1}{2}]\times[\frac{1}{2}, 1]$: Then $C(u, v) \le C(\frac{1}{2}, v) \le \xi + v - \frac{1}{2} \le v$, and hence $C(u, v) \le \min\left\{u;\ \xi + v - \frac{1}{2}\right\}$.
4. $(u, v) \in [\frac{1}{2}, a]\times[\frac{1}{2}, b]$: Then $C(u, v) \le \theta$. Furthermore, since $[u, 1]\times[v, 1]$ is a subset of $[\frac{1}{2}, 1]\times[\frac{1}{2}, 1]$ it follows from the 2-increasing property of a copula that $1 - u - v + C(u, v) \le 1 - \frac{1}{2} - \frac{1}{2} + \xi$, or equivalently, $C(u, v) \le u - \frac{1}{2} + v - \frac{1}{2} + \xi$. Hence, $C(u, v) \le \min\left\{u;\ v;\ \theta;\ u - \frac{1}{2} + v - \frac{1}{2} + \xi\right\}$.
5. $(u, v) \in [a, 1]\times[\frac{1}{2}, b]$: Then $C(u, v) \le C(u, b) \le \theta + u - a \le u$. Furthermore, since $[u, 1]\times[v, 1]$ is a subset of $[u, 1]\times[\frac{1}{2}, 1]$ it follows from the 2-increasing property of a copula that $1 - u - v + C(u, v) \le 1 - u - \frac{1}{2} + \xi + u - \frac{1}{2}$, or equivalently, $C(u, v) \le u - \frac{1}{2} + v - \frac{1}{2} + \xi$. Hence, $C(u, v) \le \min\left\{v;\ \theta + u - a;\ u - \frac{1}{2} + v - \frac{1}{2} + \xi\right\}$.
6. $(u, v) \in [\frac{1}{2}, a]\times[b, 1]$: Then $C(u, v) \le C(a, v) \le \theta + v - b \le v$. Furthermore, since $[u, 1]\times[v, 1]$ is a subset of $[\frac{1}{2}, 1]\times[v, 1]$ it follows from the 2-increasing property of a copula that $1 - u - v + C(u, v) \le 1 - \frac{1}{2} - v + \xi + v - \frac{1}{2}$, or equivalently, $C(u, v) \le u - \frac{1}{2} + v - \frac{1}{2} + \xi$. Hence, $C(u, v) \le \min\left\{u;\ \theta + v - b;\ u - \frac{1}{2} + v - \frac{1}{2} + \xi\right\}$.
7. $(u, v) \in [a, 1]\times[b, 1]$: Then since $[u, 1]\times[v, 1]$ is a subset of $[a, 1]\times[b, 1]$ it follows from the 2-increasing property of a copula that $1 - u - v + C(u, v) \le 1 - a - b + \theta$, or equivalently, $C(u, v) \le u - a + v - b + \theta$. Hence, $C(u, v) \le \min\{u;\ v;\ u - a + v - b + \theta\}$.

   In sum,

$$C(u, v) \le \overline{C}_{\beta,(a,b),\theta}(u, v)$$

$$= \begin{cases} \min\{u;\ v;\ \xi\}, & \left[0, \frac{1}{2}\right]\times\left[0, \frac{1}{2}\right]; \\ \min\left\{u - \frac{1}{2} + \xi;\ v\right\}, & \left[\frac{1}{2}, 1\right]\times\left[0, \frac{1}{2}\right]; \\ \min\left\{u;\ v - \frac{1}{2} + \xi\right\}, & \left[0, \frac{1}{2}\right]\times\left[\frac{1}{2}, 1\right]; \\ \min\left\{u;\ v;\ \theta;\ u - \frac{1}{2} + v - \frac{1}{2} + \xi\right\}, & \left[\frac{1}{2}, a\right]\times\left[\frac{1}{2}, b\right]; \\ \min\left\{u - a + \theta;\ v;\ u - \frac{1}{2} + v - \frac{1}{2} + \xi\right\}, & [a, 1]\times\left[\frac{1}{2}, b\right]; \\ \min\left\{u;\ v - b + \theta;\ u - \frac{1}{2} + v - \frac{1}{2} + \xi\right\}, & \left[\frac{1}{2}, a\right]\times[b, 1]; \\ \min\{u;\ v;\ u - a + v - b + \theta\}, & [a, 1]\times[b, 1]. \end{cases}$$

We note that $\overline{C}_{\beta,(a,b),\theta}$ is a copula (in particular, a *shuffle* of the copula M; see Nelsen (1999), pp. 59–64) if and only if $\xi + \theta \ge 1$. In this case, using the same notation as Nelsen (1999),

$$\overline{C}_{\beta,(a,b),\theta} = M\left(7, \left\{[0, \xi], \left[\xi, \frac{1}{2}\right], \left[\frac{1}{2}, 1-\xi\right], [1-\xi, \theta], [\theta, a], [a, a+b-\theta], [a+b-\theta, 1]\right\}, (1, 3, 2, 4, 6, 5, 7), 1\right).$$

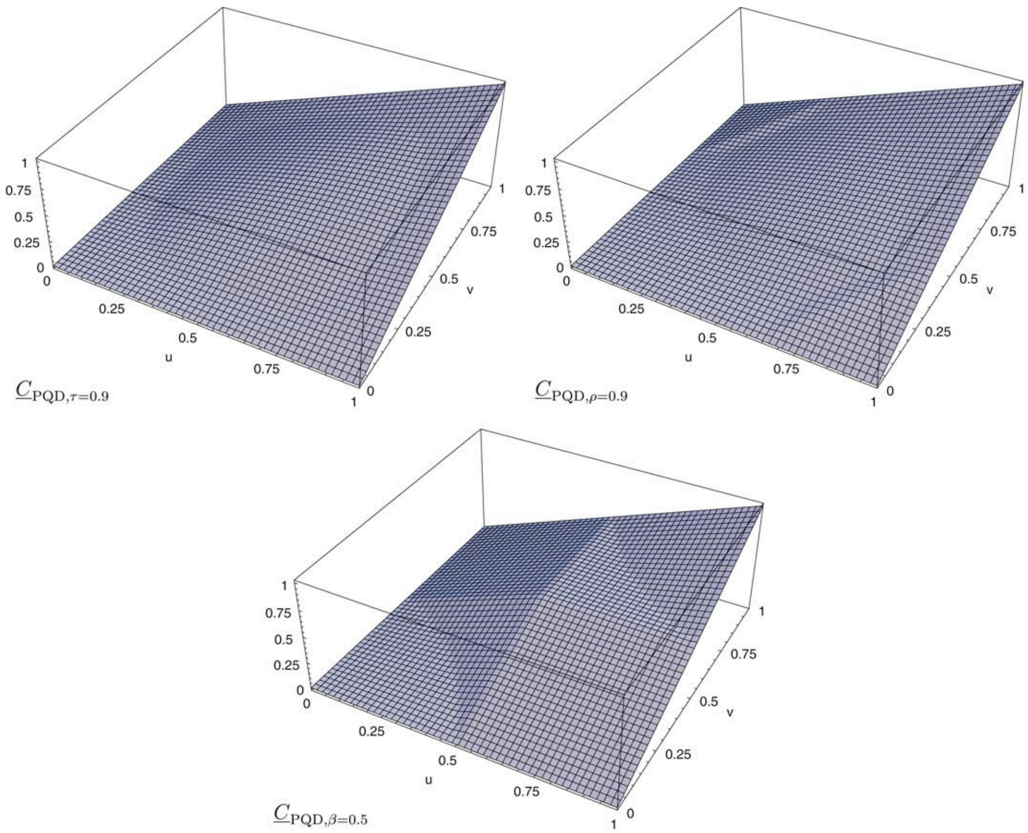

Fig. 4. Lower bounds on the copula for a given τ, ρ or β under PQD.

Notice that $\overline{C}_{\beta,(a,b),\theta}(\frac{1}{2}, \frac{1}{2}) = \xi$ and $\overline{C}_{\beta,(a,b),\theta}(a, b) = \theta$. Hence, whenever $\xi + \theta \geq 1$, it is pointwise best-possible.

Following the proof of Theorem 2 of Nelsen et al. (2001), we calculate Kendall's tau for $\overline{C}_{\beta,(a,b),\theta}$ by using the following expression, equivalent to (3) (we will not assume that $\xi + \theta \geq 1$, calculating $\tau(\overline{C}_{\beta,(a,b),\theta})$ also when $\overline{C}_{\beta,(a,b),\theta}$ is not a true copula):

$$\tau_{X,Y} = \tau(C) = 1 - 4\int_0^1 \int_0^1 \frac{\partial}{\partial u} C(u, v) \frac{\partial}{\partial v} C(u, v) \mathrm{d}u\mathrm{d}v.$$

To compute the double integral, we distinguish between the following cases of relevance:

1. $(u, v) \in [a, 1] \times [b, 1]$: Then the integrand is different from 0 in case both $u - a + b - v + \theta \leq u$ and $u - a + b - v + \theta \leq v$. Hence, we compute
$$\int_a^{a+b-\theta} \int_b^{a+b-\theta} \mathrm{d}u\mathrm{d}v = (b - \theta)(a - \theta).$$
2. $(u, v) \in [\frac{1}{2}, a] \times [\frac{1}{2}, b]$: Then the integrand is different from 0 in case $u - \frac{1}{2} + v - \frac{1}{2} + \xi \leq u$, $u - \frac{1}{2} + v - \frac{1}{2} + \xi \leq v$ and $u - \frac{1}{2} + v - \frac{1}{2} + \xi \leq \theta$. Hence, we compute
$$\int_{\frac{1}{2}}^{\min\{a;\ 1-\xi\}} \int_{\frac{1}{2}}^{\max\{\frac{1}{2};\ \min\{b;\ 1-\xi;\ 1-\xi+\theta-u\}\}} \mathrm{d}u\mathrm{d}v = \left(1 - \xi - \frac{1}{2}\right)^2.$$

Hence,

$$\tau(\overline{C}_{\beta,(a,b),\theta}) = 1 - 4\left((a - \theta)(b - \theta) + \left(1 - \xi - \frac{1}{2}\right)^2\right).$$

Notice that $C \leq \overline{C}_{\beta,(a,b),\theta}$ implies that $\tau(C) \leq \tau(\overline{C}_{\beta,(a,b),\theta})$, irrelevant of whether $\overline{C}_{\beta,(a,b),\theta}$ is a true copula. Solving (with respect to λ) for the feasible root of

$$1 - 4\left((a - \lambda)(b - \lambda) + \left(1 - \xi - \tfrac{1}{2}\right)^2\right) = \tau$$

yields

$$\lambda = \frac{1}{2}\left(a + b - \sqrt{(a - b)^2 - \tau + 4\left(\xi - \xi^2\right)}\right).$$

It follows that $\theta \geq \lambda$ and consequently that $C(a, b) \geq \max\{a + b - 1;\ \xi;\ \lambda\}$ whenever $(a, b) \in [\frac{1}{2}, 1]^2$.

To prove that (15) is pointwise best-possible whenever $\underline{C}_{\tau,\beta}(u, v) + \xi \geq 1$, we need to prove that for any $(a, b) \in [\frac{1}{2}, 1]^2$ with $\underline{C}_{\tau,\beta}(a, b) + \xi \geq 1$ there exists a copula C satisfying $\tau(C) = \tau$ and $\beta(C) = \beta$ such that $C(a, b) = \max\{a + b - 1;\ \xi;\ \lambda\}$. We distinguish between two cases. Case 1: $\lambda \geq \max\{a + b - 1;\ \xi\}$. For this case $\overline{C}_{\beta,(a,b),\lambda}$ satisfies $\tau(\overline{C}_{\beta,(a,b),\lambda}) = \tau$, $\beta(\overline{C}_{\beta,(a,b),\lambda}) = \beta$ and $\overline{C}_{\beta,(a,b),\lambda}(a, b) = \lambda$. Case 2: $\lambda < \max\{a + b - 1;\ \xi\}$. For this case $C_\alpha = \alpha\overline{C}_{\beta,(a,b),\max\{a+b-1;\ \xi\}} + (1 - \alpha)\underline{C}_\beta$, $\alpha \in [0, 1]$, is a family of copulas of which each member satisfies $\beta(C_\alpha) = \beta$ and $C_\alpha(a, b) = \max\{a + b - 1;\ \xi\}$. Furthermore, $\tau(C_\alpha)$ is a continuous function of α satisfying $\tau(\underline{C}_\beta) \leq \tau \leq \tau(\overline{C}_{\beta,(a,b),\max\{a+b-1;\ \xi\}})$. It now follows from the intermediate value theorem that there exists an $\alpha \in [0, 1]$ such that $\tau(C_\alpha) = \tau$. This completes the proof. □

The expressions for $\underline{C}_{\tau,\beta}(u, v)$ on the domains $(u, v) \in [0, \frac{1}{2}]^2$, $(u, v) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ and $(u, v) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ can be derived in a similar way. Also, using the same method of proof, a lower bound on the copula can be derived when both ρ (rather than τ) and β are given. The expressions and their derivations are not displayed in this paper to save space. The effectiveness of knowing both ρ

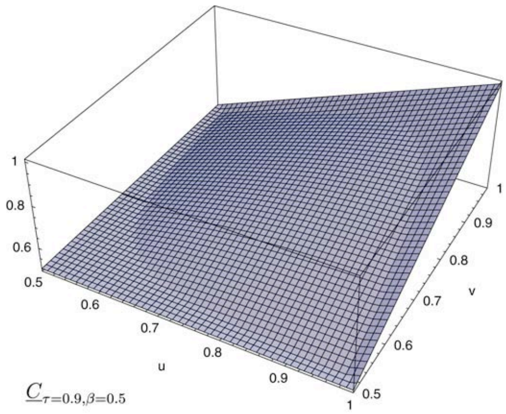

Fig. 5. Lower bounds on the copula for a given τ and β.

and β in bounding the VaR, is comparable to the situation in which both τ and β are given.

Remark 4.1 (*On $\underline{C}_{\tau,\beta}$ and Complex Values*). Notice that $\underline{C}_{\tau}(\frac{1}{2},\frac{1}{2}) \leq \xi\ (=\frac{\beta+1}{4})$ should hold for any (feasible) $\tau-\beta$ combination and that equality is attained whenever $4(\xi-\xi^2)=\tau$ holds. It means that for any (feasible) $\tau-\beta$ combination $\underline{C}_{\tau,\beta}$ is always real-valued.

Fig. 5 plots the lower bound $\underline{C}_{\tau,\beta}$ for given values of τ and β.

5. When the risks are PQD and β is given

From Theorem 3.2.2 of Nelsen (1999), one easily derives that any copula C with $C(a,b)=\theta$ for given values a, b and θ, and $C \geq \Pi$ must satisfy

$$C(u,v) \geq \begin{cases} \max\{uv;\ u-a+v-b+\theta\}, & [0,a]\times[0,b]; \\ \max\{uv;\ u-a+\theta\}, & [0,a]\times[b,1]; \\ \max\{uv;\ v-b+\theta\}, & [a,1]\times[0,b]; \\ \max\{uv;\ \theta\}, & [a,1]\times[b,1]. \end{cases} \quad (16)$$

We note that the function on the RHS of (16) is not in general a copula as it is not 2-increasing, i.e., it does not satisfy condition (1) (consider for example $a=0.5$, $b=0.5$, $\theta=0.36$, $u_1=0.36$, $u_2=0.4$, $v_1=0.6$ and $v_2=0.65$). Also, $\min\{v;\ \max\{uv;\ u+a-\theta\}\}$ and $\min\{u;\ \max\{uv;\ v+b-\theta\}\}$, though satisfying the boundary conditions, are not copulas as they are not 2-increasing.

The question arises whether the lower bound in (16) is pointwise best-possible. We state the following theorem, answering this question for the case that $a=\frac{1}{2}$ and $b=\frac{1}{2}$:

Theorem 5.1. *Let C be a copula and suppose that $\beta(C)=\beta\geq 0$ and that $C\geq\Pi$. Then $C(u,v)\geq \underline{C}_{\mathrm{PQD},\beta}(u,v)$, for all $(u,v)\in[0,1]^2$, with*

$$\underline{C}_{\mathrm{PQD},\beta}(u,v) = \max\left\{uv;\ \frac{\beta+1}{4}-\left(\frac{1}{2}-u\right)_+-\left(\frac{1}{2}-v\right)_+\right\}. \quad (17)$$

The bound $\underline{C}_{\mathrm{PQD},\beta}$ is pointwise best-possible whenever $\underline{C}_{\mathrm{PQD},\beta}(u,v)+\frac{\beta+1}{4}\geq 1$, $(u,v)\in[\frac{1}{2},1]^2$.

Proof. Substituting $a=\frac{1}{2}$, $b=\frac{1}{2}$ and $\theta=\frac{\beta+1}{4}$ in the RHS of (16) readily yields the lower bound (17).

To prove that (17) is pointwise best-possible whenever $\underline{C}_{\mathrm{PQD},\beta}(u,v)+\frac{\beta+1}{4}\geq 1$, $(u,v)\in[\frac{1}{2},1]^2$, we need to prove that for any $(a,b)\in[\frac{1}{2},1]^2$ there exists a copula C satisfying $\beta(C)=\beta\geq 0$ and $C\geq\Pi$ such that $C(a,b)=\max\left\{ab;\ \frac{\beta+1}{4}-\left(\frac{1}{2}-a\right)_+-\left(\frac{1}{2}-b\right)_+\right\}$. Recall that $\underline{C}_{\mathrm{PQD},\beta}$ is not in general a copula and therefore, though satisfying $\beta(\underline{C}_{\mathrm{PQD},\beta})=\beta$ and $\underline{C}_{\mathrm{PQD},\beta}\geq\Pi$, should not be used to prove sharpness of the bound. However, one easily verifies that the copula $\overline{C}_{\beta,(a,b),\theta}$ defined in the proof of Theorem 4.1 satisfies these conditions whenever

$$\theta=\max\left\{ab;\ \frac{\beta+1}{4}-\left(\frac{1}{2}-a\right)_+-\left(\frac{1}{2}-b\right)_+\right\}.$$

This proves the stated result. $\square$

We note that $\underline{C}_{\mathrm{PQD},\beta}$ is an improvement over Π for any $\beta>0$. Fig. 4 plots the lower bound $\underline{C}_{\mathrm{PQD},\beta}$ for a given value of β.

6. Worst VaR scenarios

In this section, we derive explicit upper and lower bounds on the VaR under various sets of assumptions on the marginals, the measures of association and other dependence properties. We then assess the effectiveness in bounding the VaR for the different sources of dependence information. We restrict attention to $\psi(x,y)=x+y$.

6.1. Uniform marginals

6.1.1. When τ, ρ or β is given

Let F_1 and F_2 be uniform (0, 1) laws. Substitution of $\underline{C}_\tau$, $\underline{C}_\rho$ and $\underline{C}_\beta$ ((6)–(8)) in (10) and (11) yields the following inequalities:

$$\max\left\{0;\ s-1;\ \frac{1}{2}\left(s-\sqrt{1-\tau_{U,V}}\right)\right\}\leq\mathbb{P}[U+V<s]$$
$$\leq\mathbb{P}[U+V\leq s]\leq\min\left\{1;\ s;\ \frac{1}{2}\left(s+\sqrt{1-\tau_{U,V}}\right)\right\};$$
$$\max\left\{0;\ s-1;\ \frac{1}{2}s-\sqrt[3]{\frac{1-\rho_{U,V}}{12}}\right\}\leq\mathbb{P}[U+V<s]$$
$$\leq\mathbb{P}[U+V\leq s]\leq\min\left\{1;\ s;\ \frac{1}{2}s+\sqrt[3]{\frac{1-\rho_{U,V}}{12}}\right\};$$
$$\max\left\{0;\ s-1;\ \frac{\beta_{U,V}+1}{4}-(1-s)_+\right\}\leq\mathbb{P}[U+V<s]$$
$$\leq\mathbb{P}[U+V\leq s]\leq\min\left\{1;\ s;\ 1-\frac{\beta_{U,V}+1}{4}+(s-1)_+\right\}.$$

Here U and V are two uniform (0, 1) r.v.'s having a measure of association of $\tau_{U,V}$, $\rho_{U,V}$ and $\beta_{U,V}$, respectively. Hence, we find that

$$\left.\begin{matrix} p, & p\in(0,\sqrt{1-\tau_{U,V}}]; \\ 2p-\sqrt{1-\tau_{U,V}}, & p\in(\sqrt{1-\tau_{U,V}},1]. \end{matrix}\right\}\leq\mathrm{VaR}_p[U+V]$$
$$\leq\begin{cases} 2p+\sqrt{1-\tau_{U,V}}, & p\in(0,1-\sqrt{1-\tau_{U,V}}]; \\ 1+p, & p\in(1-\sqrt{1-\tau_{U,V}},1]. \end{cases}$$
$$\left.\begin{matrix} p, & p\in\left(0,2\sqrt[3]{\frac{1-\rho_{U,V}}{12}}\right]; \\ 2p-2\sqrt[3]{\frac{1-\rho_{U,V}}{12}}, & p\in\left(2\sqrt[3]{\frac{1-\rho_{U,V}}{12}},1\right]. \end{matrix}\right\}\leq\mathrm{VaR}_p[U+V]$$
$$\leq\begin{cases} 2p+2\sqrt[3]{\frac{1-\rho_{U,V}}{12}}, & p\in\left(0,1-2\sqrt[3]{\frac{1-\rho_{U,V}}{12}}\right]; \\ 1+p, & p\in\left(1-2\sqrt[3]{\frac{1-\rho_{U,V}}{12}},1\right]. \end{cases}$$

$$\left.\begin{array}{ll} p, & p \in \left(0, 1 - \dfrac{\beta_{U,V}+1}{4}\right]; \\ p + \dfrac{\beta_{U,V}+1}{4}, & p \in \left(1 - \dfrac{\beta_{U,V}+1}{4}, 1\right]. \end{array}\right\} \le \mathrm{VaR}_p[U+V]$$
$$\le \begin{cases} 1 + p - \dfrac{\beta_{U,V}+1}{4}, & p \in \left(0, \dfrac{\beta_{U,V}+1}{4}\right]; \\ 1 + p, & p \in \left(\dfrac{\beta_{U,V}+1}{4}, 1\right]. \end{cases}$$

6.1.2. When the risks are PQD and τ, ρ or β is given

Substitution of $\underline{C}_{\mathrm{PQD},\tau}$, $\underline{C}_{\mathrm{PQD},\rho}$ and $\underline{C}_{\mathrm{PQD},\beta}$ ((13), (14) and (17)) in (10) and (11) yields the following inequalities:

$$\max\left\{\frac{1}{4}s^2;\ \frac{1}{2}\left(s - \sqrt{1-\tau_{U,V}}\right)\right\} \le \mathbb{P}[U+V<s]$$
$$\le \mathbb{P}[U+V\le s]$$
$$\le \min\left\{s - \frac{1}{4}s^2;\ \frac{1}{2}\left(s + \sqrt{1-\tau_{U,V}}\right)\right\};$$
$$\max\left\{\frac{1}{4}s^2;\ \frac{1}{2}s - \sqrt[3]{\frac{1-\rho_{U,V}}{12}}\right\} \le \mathbb{P}[U+V<s]$$
$$\le \mathbb{P}[U+V\le s]$$
$$\le \min\left\{s - \frac{1}{4}s^2;\ \frac{1}{2}s + \sqrt[3]{\frac{1-\rho_{U,V}}{12}}\right\};$$
$$\max\left\{\frac{1}{4}s^2;\ \frac{\beta+1}{4} - (1-s)_+\right\} \le \mathbb{P}[U+V<s]$$
$$\le \mathbb{P}[U+V\le s]$$
$$\le \min\left\{s - \frac{1}{4}s^2;\ 1 - \frac{\beta+1}{4} + (s-1)_+\right\}.$$

Here U and V are two uniform $(0,1)$ and PQD r.v.'s having a measure of association of $\tau_{U,V} \ge 3/4$, $\rho_{U,V} \ge 13/16$ and $\beta_{U,V} \ge 0$, respectively. Hence, we find that

$$\max\left\{2(1-\sqrt{1-p});\ 2p - \sqrt{1-\tau_{U,V}}\right\} \le \mathrm{VaR}_p[U+V]$$
$$\le \min\left\{2p + \sqrt{1-\tau_{U,V}};\ 2\sqrt{p}\right\};$$
$$\max\left\{2(1-\sqrt{1-p});\ 2p - 2\sqrt[3]{\frac{1-\rho_{U,V}}{12}}\right\} \le \mathrm{VaR}_p[U+V]$$
$$\le \min\left\{2p + 2\sqrt[3]{\frac{1-\rho_{U,V}}{12}};\ 2\sqrt{p}\right\};$$
$$\max\left\{2(1-\sqrt{1-p});\ \left(p + \frac{\beta_{U,V}+1}{4}\right)\mathbb{I}_{\{p\in(1-\frac{\beta_{U,V}+1}{4},1]\}}\right\}$$
$$\le \mathrm{VaR}_p[U+V]$$
$$\le \min\left\{\left(1 + p - \frac{\beta_{U,V}+1}{4}\right)\mathbb{J}_{\{p\in(0,\frac{\beta_{U,V}+1}{4}]\}};\ 2\sqrt{p}\right\}.$$

Here $\mathbb{I}_A$ denotes the indicator function of event A, taking the value 1 if A is true and 0 otherwise; and $\mathbb{J}_A$ takes the value 1 if A is true and $+\infty$ otherwise.

6.1.3. When both τ and β are given

Let $s \ge \frac{3}{2}$. Substitution of $\underline{C}_{\tau,\beta}$ (15) in (10) and (11) yields the following inequalities:

$$\max\left\{s-1;\ \frac{1}{2}\left(s - \sqrt{4(\xi_{U,V} - \xi_{U,V}^2) - \tau_{U,V}}\right)\right\}$$

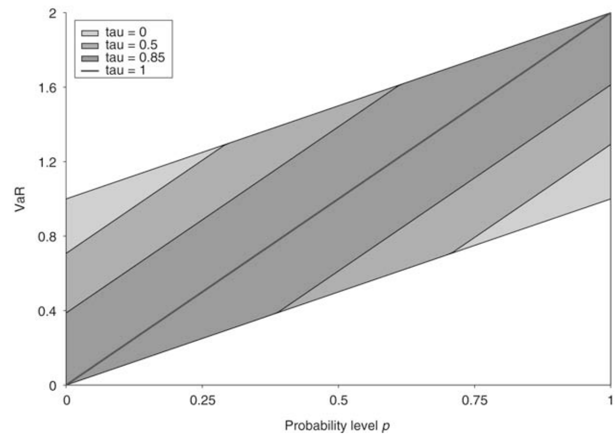

Fig. 6. Bounds on the VaR for a given τ and uniform $(0, 1)$ marginals.

$$\le \mathbb{P}[U+V<s]$$
$$\le \mathbb{P}[U+V\le s]$$
$$\le \min\left\{1;\ \frac{1}{2}\left(s + \sqrt{4(\xi_{U,V} - \xi_{U,V}^2) - \tau_{U,V}}\right)\right\}.$$

Here U and V are two uniform $(0,1)$ r.v.'s having a measure of association of $\tau_{U,V}$ and $\beta_{U,V} = 4\xi_{U,V} - 1$. Hence, we find that

$$\left.2p - \sqrt{4(\xi_{U,V} - \xi_{U,V}^2) - \tau_{U,V}}, \quad p \in (p^*, 1];\right\}$$
$$\le \mathrm{VaR}_p[U+V]$$
$$\le \begin{cases} 2p + \sqrt{4(\xi_{U,V} - \xi_{U,V}^2) - \tau_{U,V}}, \\ \quad p \in (p^{**}, 1 - \sqrt{4(\xi_{U,V} - \xi_{U,V}^2) - \tau_{U,V}}]; \\ 1 + p, \\ \quad p \in (p^{**} \vee (1 - \sqrt{4(\xi_{U,V} - \xi_{U,V}^2) - \tau_{U,V}}), 1]. \end{cases}$$

Here, $p^* = \min\left\{1;\ \frac{1}{2}\left(\frac{3}{2} + \sqrt{4(\xi_{U,V} - \xi_{U,V}^2) - \tau_{U,V}}\right)\right\}$ and $p^{**} = \max\left\{\frac{1}{2};\ \frac{1}{2}\left(\frac{3}{2} - \sqrt{4(\xi_{U,V} - \xi_{U,V}^2) - \tau_{U,V}}\right)\right\}$.

6.2. Lognormal marginals

We now replace the uniform $(0, 1)$ marginals by lognormal $(0, 1)$ marginals and repeat the substitution and inversion operations of Sections 6.1.1 and 6.1.2. To save space we do not display the obtained expressions for the bounds, but only include the corresponding graphs.

6.3. Effectiveness

The effectiveness of different types of nonparametric information on the dependence structure in bounding the VaR is easiest studied on the basis of graphical illustrations; see Figs. 6–14. In the graphs, darker colored areas are piled on top of lighter colored areas so that lighter colored areas are effectively included in darker colored areas.

6.3.1. When τ, ρ or β is given

Notice that for probability levels relevant in risk pricing and risk management ($p \ge 0.75$, say) the upper bound on the VaR is improved only for large values of τ or ρ. Furthermore, notice that information on β appears to be useless for this purpose. On the contrary, the lower bound on the VaR when the probability level tends to 1 is improved for all $\tau > 0$, $\rho > -0.5$ and $\beta > -1$.

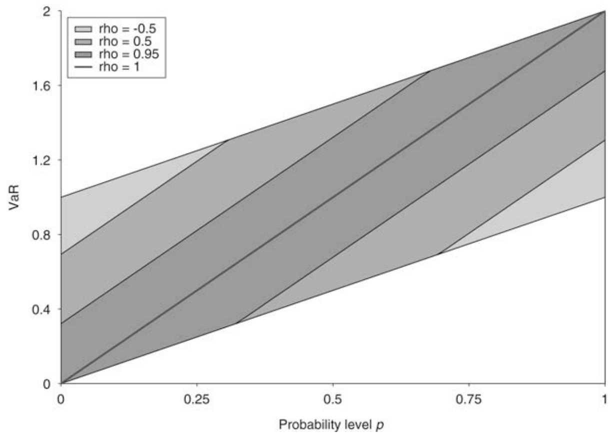

Fig. 7. Bounds on the VaR for a given ρ and uniform (0, 1) marginals.

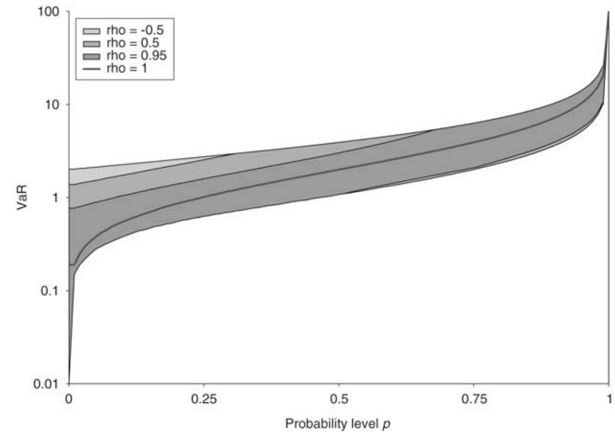

Fig. 10. Bounds on the VaR for a given ρ and lognormal (0, 1) marginals.

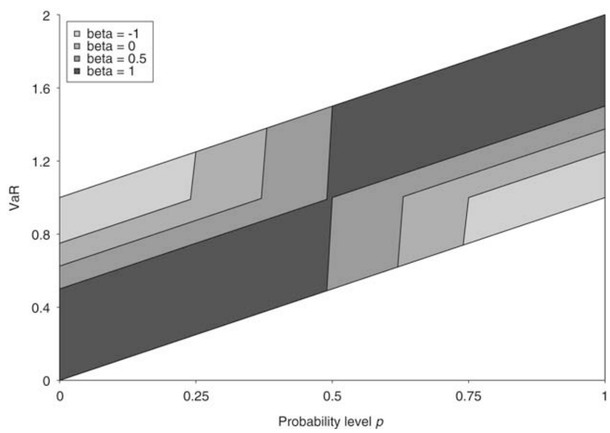

Fig. 8. Bounds on the VaR for a given β and uniform (0, 1) marginals.

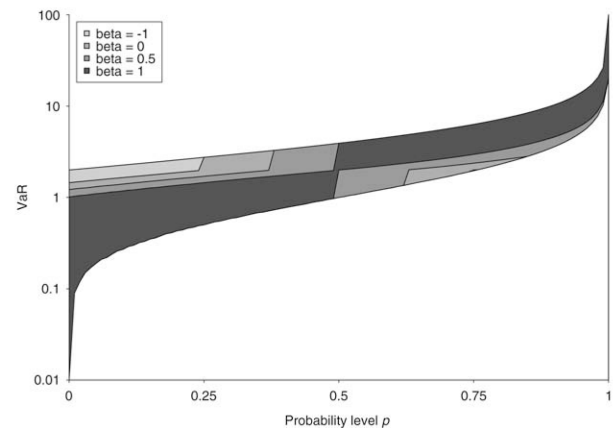

Fig. 11. Bounds on the VaR for a given β and lognormal (0, 1) marginals.

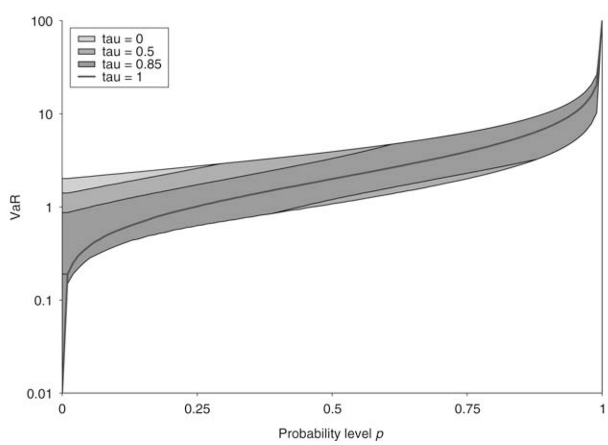

Fig. 9. Bounds on the VaR for a given τ and lognormal (0, 1) marginals.

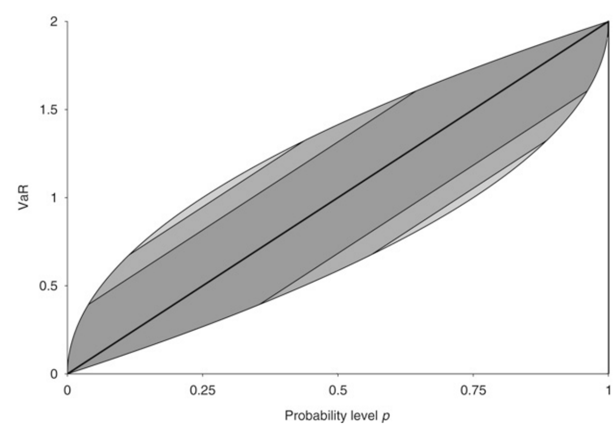

Fig. 12. Bounds on the VaR for a given τ and uniform (0, 1) and PQD marginals; $\tau = 0.75, 0.8, 0.9, 1$ (from light to dark).

It is perhaps slightly counterintuitive that the upper bound on the VaR decreases when τ, ρ or β increases. This can be explained by the fact that the larger the τ, ρ or β the larger the "distance" between the (unimproved) lower bound W and $\underline{C}_\tau$, $\underline{C}_\rho$ and $\underline{C}_\beta$, respectively.

6.3.2. When the risks are PQD and τ, ρ or β is given

From a comparison between Figs. 6–8 and 12–14, it becomes apparent that the information that the risks are PQD is rather effective in bounding the VaR, both from above and from below. For probability levels relevant in risk pricing and risk management, additional information on τ or ρ leads to improved upper bounds only when τ or ρ are large. Information on β seems to be useless for this purpose (but recall that because $\underline{C}_{\mathrm{PQD},\beta}$ is not in general a copula, the corresponding VaR bounds may fail to be sharp). The lower bound on the VaR when the probability level tends to 1 is improved for most values of $\tau > 3/4$, $\rho > 13/16$ and $\beta > 0$. We find again that the upper bound on the VaR decreases when τ, ρ or β increases.

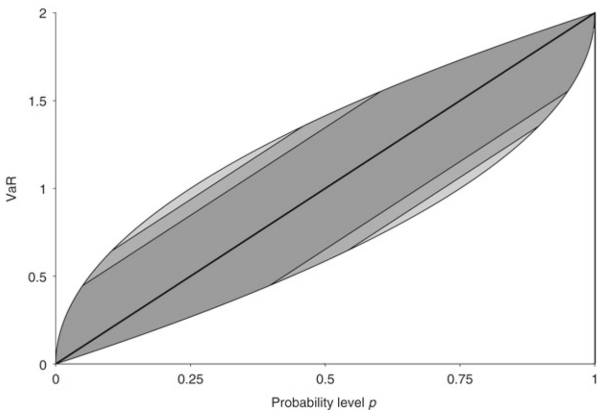

Fig. 13. Bounds on the VaR for a given ρ and uniform (0, 1) and PQD marginals; $\rho = 13/16, 14/16, 15/16, 1$ (from light to dark).

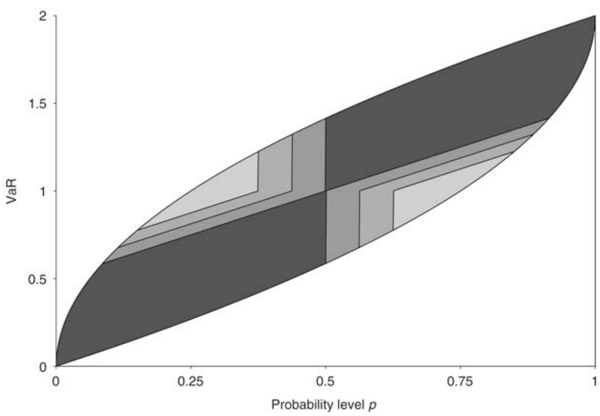

Fig. 14. Bounds on the VaR for a given β and uniform (0, 1) and PQD marginals $\beta = 0, 0.5, 0.75, 1$ (from light to dark).

6.3.3. *When both τ and β are given*

Depending on the particular values of the measures of association τ and β, the upper and lower bounds on the VaR may be well improved for probability levels relevant in risk pricing and risk management, with optimal effectiveness whenever $4(\xi - \xi^2) = \tau$ holds. This means that knowing both τ and β could conceivably be significantly more informative in this context than knowing only τ or β.

7. Worst TVaR scenarios

We state the following definition:

Definition 7.1. The Tail-Value-at-Risk (TVaR) at probability level $p \in [0, 1)$ for a r.v. Z with d.f. G is defined as

$$\mathrm{TVaR}_p[Z] = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q[Z]\mathrm{d}q. \tag{18}$$

Equivalently,

$$\begin{aligned}\mathrm{TVaR}_p[Z] &= \mathrm{VaR}_p[Z] + \frac{1}{1-p}\mathbb{E}[(Z - \mathrm{VaR}_p[Z])_+] \\ &= \inf_{d\in\mathbb{R}}\left\{d + \frac{1}{1-p}\mathbb{E}[(Z-d)_+]\right\},\end{aligned} \tag{19}$$

where, as usual, $\inf\{\emptyset\} = +\infty$ by convention. Notice that TVaR_p is a continuous function of p. It is the arithmetic average of $\mathrm{VaR}_q[Z]$, $q \in (p, 1]$.

A proof of the second equality in (19) is contained in Laeven and Goovaerts (2004); see also the geometric proof of Dhaene et al. (2008). Finding best-possible upper and lower bounds on the TVaR is easier than finding best-possible upper and lower bounds on the VaR. To show why this is so, we first introduce the notion of *supermodularity*:

Definition 7.2. A function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is supermodular if for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$\begin{aligned}\varphi(x_1, y_1) + \varphi(x_2, y_2) \le\ &\varphi(x_1 \wedge x_2, y_1 \wedge y_2) \\ &+ \varphi(x_1 \vee x_2, y_1 \vee y_2).\end{aligned}$$

Then we state the following result:

Lemma 7.1. *Suppose that there exist copulas $\underline{C}$ and $\overline{C}$ such that*

$$\underline{C}(F_1(x), F_2(y)) \le F(x, y) \le \overline{C}(F_1(x), F_2(y)), \tag{20}$$

for all $(x, y) \in \mathbb{R}^2$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function that is supermodular. Furthermore, let $(X^{\underline{C}}, Y^{\underline{C}})$ and $(X^{\overline{C}}, Y^{\overline{C}})$ denote two random vectors with bivariate d.f. $\underline{C}(F_1, F_2)$ and $\overline{C}(F_1, F_2)$, respectively. Then

$$\begin{aligned}\mathrm{TVaR}_p[\varphi(X^{\underline{C}}, Y^{\underline{C}})] &\le \mathrm{TVaR}_p[\varphi(X, Y)] \\ &\le \mathrm{TVaR}_p[\varphi(X^{\overline{C}}, Y^{\overline{C}})], \quad p \in [0, 1),\end{aligned} \tag{21}$$

provided that the TVaR's exist.

Proof. By Proposition 6.2.5 of Denuit et al. (2005), (20) is equivalent to

$$\mathbb{E}[\varphi(X^{\underline{C}}, Y^{\underline{C}})] \le \mathbb{E}[\varphi(X, Y)] \le \mathbb{E}[\varphi(X^{\overline{C}}, Y^{\overline{C}})],$$

for all measurable functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ that are supermodular, provided that the expectations exist.

Let the function $f : \mathbb{R} \to \mathbb{R}$ be non-decreasing and convex. Then (see e.g., Proposition 3.4.67 of Denuit et al. (2005)) $f \circ \varphi$ is supermodular too. It follows that,

$$\begin{aligned}\mathbb{E}[(\varphi(X^{\underline{C}}, Y^{\underline{C}}) - d)_+] &\le \mathbb{E}[(\varphi(X, Y) - d)_+] \\ &\le \mathbb{E}[(\varphi(X^{\overline{C}}, Y^{\overline{C}}) - d)_+], \quad d \in \mathbb{R},\end{aligned}$$

provided that the expectations exist. This means that $\varphi(X^{\underline{C}}, Y^{\underline{C}})$, $\varphi(X, Y)$ and $\varphi(X^{\overline{C}}, Y^{\overline{C}})$ satisfy a stop-loss order relation (i.e., they have ordered stop-loss premiums). Stop-loss order induces ordered TVaR's; see e.g., Proposition 3.4.7 (ii) of Denuit et al. (2005). This completes the proof. □

Remark 7.1. In contrast to the VaR, for which upper and lower bounds are both determined by a lower bound on the copula, upper and lower bounds for the TVaR are determined by an upper and a lower bound on the copula, respectively.

7.1. *Uniform marginals*

7.1.1. *When τ is given*

Upper and lower bounds on the TVaR will be determined using simulation. To simulate from $\overline{C}_\tau$ (see (11) of Nelsen et al. (2001)) and $\underline{C}_\tau$ the following algorithm is adopted:

1. Simulate U and Z from a uniform (0, 1) distribution (with U and Z independent).
2. Calculate $C(v|u) = \frac{\partial C(u,v)}{\partial u}$ and its generalized inverse.
3. Set $V = C^{-1}(Z|U)$. Then (U, V) is a drawing from C.

To calculate the corresponding $\mathrm{TVaR}_p[U + V]$ we proceed as follows:

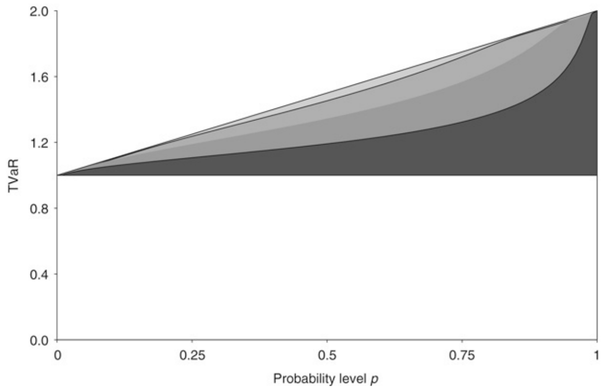

Fig. 15. Bounds on the TVaR for a given τ and uniform (0, 1) marginals; $\tau = 0$, -0.5, -0.8, -0.95 (from light to dark).

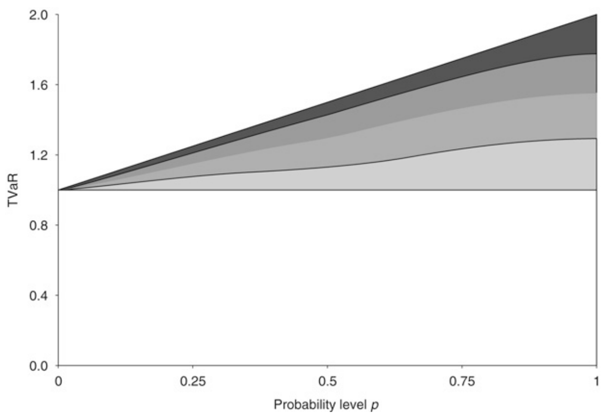

Fig. 16. Bounds on the TVaR for a given τ and uniform (0, 1) marginals; $\tau = 0$, 0.5, 0.8, 0.95 (from light to dark).

1. Set $S = U + V$.
2. Calculate the empirical d.f. of S and its generalized inverse (i.e., the VaR of S).
3. Finally, calculate the TVaR by approximating the integral in (18).

The number of simulations carried out is 100,000. The results are displayed in Figs. 15 and 16.

7.2. Effectiveness

The upper bound on the TVaR is improved whenever $\tau < 0$ (or $\rho < 1/2$). On the contrary, the lower bound is improved whenever $\tau > 0$ (or $\rho > -1/2$). In contrast to the VaR, both the lower and the upper bound on the TVaR increase when τ, ρ or β increases.

The effectiveness of other types of nonparametric dependence information in bounding the TVaR can be analyzed in a similar way.

7.3. Comparison between VaR and TVaR

We summarize below in a few statements the aggregation properties of the VaR and the TVaR. We recall that both the VaR and the TVaR are additive for sums of comonotonic r.v.'s. As the reader may verify from the conducted analysis (and its graphical illustrations):

- The comonotonic dependence structure (implying $\tau = \rho = \beta = 1$) does not give rise to the worst VaR scenario.
- The VaR is neither always subadditive nor always superadditive.
- The worst TVaR scenario is when the risks are comonotonic.
- The TVaR is always subadditive.
- The lower bound on the VaR is monotonically increasing in the value of τ, ρ or β while the upper bound on the VaR is monotonically decreasing in the value of τ, ρ or β.

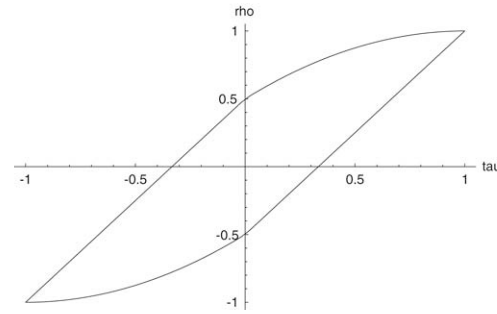

Fig. 17. τ–ρ region.

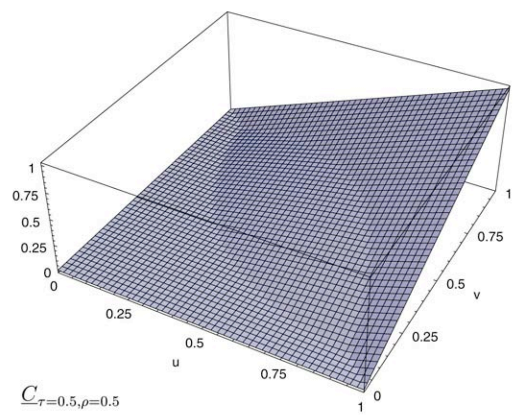

Fig. 18. Lower bound on the copula when both τ and ρ are given.

- The bounds on the TVaR are both monotonically increasing in the value of τ, ρ or β.
- For probability levels relevant in risk pricing and risk management, the upper bound on the VaR is improved only for large values of τ or ρ.
- The upper bound on the TVaR is improved whenever $\tau < 0$ or $\rho < \frac{1}{2}$.

8. An open problem: When both τ and ρ are given

Let X and Y be continuous r.v.'s. Then

$$\frac{3\tau_{X,Y} - 1}{2} \leq \rho_{X,Y} \leq \frac{1 + 2\tau_{X,Y} - \tau_{X,Y}^2}{2}, \quad \tau_{X,Y} \geq 0;$$
$$\frac{\tau_{X,Y}^2 + 2\tau_{X,Y} - 1}{2} \leq \rho_{X,Y} \leq \frac{1 + 3\tau_{X,Y}}{2}, \quad \tau_{X,Y} \leq 0. \tag{22}$$

Fig. 17 plots the τ–ρ region. The linear portion of the boundary of the τ–ρ region cannot be improved. It may be possible however to improve the parabolic portion of the boundary of the τ–ρ region; see Nelsen (1999), pp. 144–146).

From (6) and (7) one easily verifies that for a given (feasible) τ–ρ combination a copula C with $\tau(C) = \tau$ and $\rho(C) = \rho$ must satisfy

$$C(u, v) \geq \max\left\{0;\ u + v - 1;\ \frac{1}{2}\left(u + v - \sqrt{(u - v)^2 + 1 - \tau}\right);\ \frac{1}{2}\left(u + v - \phi(u, v, \rho)\right)\right\},$$

for all $(u, v) \in [0, 1]^2$. Fig. 18 plots the lower bound $\underline{C}_{\tau,\rho}$ for given values of τ and ρ.

The question arises whether this lower bound is pointwise best-possible, i.e., can we find for arbitrarily given values $(a, b) \in [0, 1]^2$ and a (feasible) τ–ρ combination a copula C with $\tau(C) = \tau$ and $\rho(C) = \rho$ such that

$$C(a, b) = \max\left\{0;\ a+b-1;\ \frac{1}{2}\left(a+b-\sqrt{(a-b)^2+1-\tau}\right);\ \frac{1}{2}\left(a+b-\phi(a,b,\rho)\right)\right\}.$$

Let us write $\frac{1}{2}\left(a+b-\sqrt{(a-b)^2+1-\tau}\right) = \lambda_1$ and $\frac{1}{2}(a+b-\phi(a,b,\rho)) = \lambda_2$. Assuming that $\lambda_1, \lambda_2 \geq \max\{0;\ a+b-1\}$, we consider the following three cases:

1. $\lambda_1 = \lambda_2$: in this case the problem is trivial;
2. $\lambda_1 > \lambda_2$: in this case the authors can find a copula C with $\tau(C) = \tau$ and $C(a, b) = \lambda_1$ but with $\rho(C) > \rho$;
3. $\lambda_1 < \lambda_2$: in this case the authors can find a copula C with $\rho(C) = \rho$ and $C(a, b) = \lambda_2$ but with $\tau(C) > \tau$.

9. Concluding remarks

Our analysis has demonstrated that even when the marginal distributions are given and the values of some commonly used nonparametric dependence measures are known, the Value-at-Risk for a function of two random variables may still vary widely. It warns the reader against using multivariate inference techniques that *implicitly* suppose that this is not so.

From our analysis it becomes explicit that the dependence measures Kendall's tau, Spearman's rho and Blomqvist's beta, which effectively measure dependence in the central part of the bivariate distribution, contain little probabilistic information for the purpose of bounding the Value-at-Risk for probability levels relevant in risk pricing and risk management. It motivates the analysis and use of appropriate tail dependence measures.

The information that the risks are positively quadrant dependent is rather effective in bounding the Value-at-Risk, both from above and from below. Joint information on both Kendall's tau and Blomqvist's beta may well improve the upper and lower bounds on the Value-at-Risk when compared with the situation in which only one of the two measures of association is given.

We have assumed throughout that the marginal distributions are fully specified. In the case where only partial information (e.g., several moments) on the marginal distributions is available, best-possible lower bounds on $\mathbb{P}[\psi(X, Y) < t]$ can be obtained similarly, by using pointwise best-possible lower bounds on the marginal distributions; see Section 8.4.8 of Denuit et al. (2005) for related results.

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References

Denuit, Michel, Genest, Christian, Marceau, Étienne, 1999. Stochastic bounds on sums of dependent risks. Insurance: Mathematics and Economics 25, 85–104.
Denuit, Michel, Scaillet, Olivier, 2004. Nonparametric tests for positive quadrant dependence. Journal of Financial Econometrics 2, 422–450.
Denuit, Michel, Dhaene, Jan, Goovaerts, Marc J., Kaas, Rob, 2005. Actuarial Theory for Dependent Risks. Wiley, New York.
Dhaene, Jan, Goovaerts, Marc J., 1996. Dependency of risks and stop-loss order. Astin Bulletin 26, 201–212.
Dhaene, Jan, Denuit, Michel, Goovaerts, Marc J., Kaas, Rob, Vyncke, David, 2002. The concept of comonotonicity in actuarial science and finance: Theory. Insurance: Mathematics and Economics 31, 3–33.
Dhaene, Jan, Laeven, Roger J.A., Vanduffel, Steven, Darkiewicz, Gregory, Goovaerts, Marc J., 2008. Can a coherent risk measure be too subadditive? Journal of Risk and Insurance 75, 365–386.
Embrechts, Paul, Höing, Andrea, Juri, Alessandro, 2003. Using copulae to bound the Value-at-Risk for functions of dependent risks. Finance and Stochastics 7, 145–167.
Embrechts, Paul, Höing, Andrea, Puccetti, Giovanni, 2005. Worst VaR Scenarios. Insurance: Mathematics and Economics 37, 115–134.
Embrechts, Paul, Puccetti, Giovanni, 2006. Bounds for functions of dependent risks. Finance and Stochastics 10, 341–352.
Frank, Maurice J., Nelsen, Roger B., Schweizer, Berthold, 1987. Best-possible bounds for the distribution of a sum — A problem of Kolmogorov. Probability Theory and Related Fields 74, 199–211.
Laeven, Roger J.A., Goovaerts, Marc J., 2004. An optimization approach to the dynamic allocation of economic capital. Insurance: Mathematics and Economics 35, 299–319.
Laeven, Roger J.A, 2009. Worst VaR scenarios: A remark. Insurance: Mathematics and Economics 44 (2), 159–163.
Makarov, G.D., 1981. Estimates for the distribution function of the sum of two random variables with given marginal distributions. Theory of Probability and its Applications 26, 815–817.
Nelsen, Roger B., 1999. An Introduction to Copulas. Springer, New York.
Nelsen, Roger B., Quesada Molina, José Juan, Rodríguez Lallena, José Antonio, Úbeda Flores, Manuel, 2001. Bounds on bivariate distribution functions with given margins and measures of association. Communications in Statistics -Theory and Methods 30 (6), 1155–1162.
Nelsen, Roger B., Quesada Molina, José Juan, Rodríguez Lallena, José Antonio, Úbeda Flores, Manuel, 2004. Best-possible bounds on sets of bivariate distribution functions. Journal of Multivariate Analysis 90, 348–358.
Nelsen, Roger B., Úbeda Flores, Manuel, 2004. A comparison of bounds on sets of joint distribution functions derived from various measures of association. Communications in Statistics -Theory and Methods 33 (10), 2299–2305.
Rüschendorf, Ludger, 1982. Random variables with maximum sums. Advances in Applied Probability 14, 623–632.
Sklar, Abe, 1959. Fonctions de répartition à *n* dimensions et leurs marges. Publications de l'Institut de Statistique de l'Université de Paris 8, 229–231.
Wang, Shaun S., Dhaene, Jan, 1998. Comonotonicity, correlation order and premium principles. Insurance: Mathematics and Economics 22, 235–242.
Williamson, R.C., Downs, T., 1990. Probabilistic arithmetic I: Numerical methods for calculating convolutions and dependency bounds. International Journal of Approximate Reasoning 4, 89–158.