Teaching Tip: Partial Sums of Geometric Series

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In nearly every modern calculus text the standard method of evaluating the *n*th partial sum S_n of the geometric series with first term *a* and common ratio *r* (for $r \neq 1$) goes as follows:

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^{n-1}, \\ rS_n &= \phantom{a + {}} ar + ar^2 + \cdots + ar^{n-1} + ar^n, \\ (1-r)S_n &= a \phantom{+ ar + ar^2 + \cdots + ar^{n-1}} - ar^n, \\ \therefore S_n &= a\frac{1-r^n}{1-r}. \end{aligned}$$

The difficulty that some students may experience with this argument is "seeing" the cancellation of the terms represented by the ellipses. The use of ellipses can be avoided by using two equivalent forms for S_{n+1}:

$$S_{n+1} = S_n + ar^n = a + rS_n,$$

so that $(1-r)S_n = a(1-r^n)$ and the desired result follows.

This method of expressing the (*n*+1)st partial sum in two ways can be applied to other series. For example, consider the "differentiated" geometric series $a + 2ar + 3ar^2 + \cdots + nar^{n-1} + \cdots$. If we denote its *n*th partial sum by T_n, then we have

$$\begin{aligned} T_{n+1} &= a + 2ar + 3ar^2 + \cdots + nar^{n-1} + (n+1)ar^n, \\ &= T_n + (n+1)ar^n = rT_n + (a + ar + ar^2 + \cdots + ar^n), \end{aligned}$$

and hence $(1-r)T_n = S_{n+1} - (n+1)ar^n$, from which it follows (for $r \neq 1$) that

$$T_n = \frac{a}{(1-r)^2}\left[1 - (n+1)r^n + nr^{n+1}\right].$$