Concordance and copulas: A survey

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Abstract.

In this paper we survey relationships between concordance of random variables and their copulas. We focus on the relationship between concordance and measures of association such as Kendall's tau, Spearman's rho and Gini's coefficient. Extensions to the multivariate case are also discussed.

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1. Introduction

There are a variety of ways to describe and to measure dependence between random variables. Many such descriptions and measures are, in the words of Hoeffding (1940), "scale-invariant", that is, they remain unchanged under strictly increasing transformations of the random variables. An important scale-invariant notion is concordance.

As Schweizer and Wolff (1981) note, "...it is precisely the copula which captures those properties of the joint distribution which are invariant under almost surely strictly increasing transformations". The term copula, coined by Sklar (1959), is now common in the statistical literature; we assume that the reader is familiar with copulas and their basic properties.

As a consequence of Theorem 3 in Schweizer and Wolff (1981) scale-invariant properties and measures are expressible in terms of the copula of the random variables. The focus of this paper is a survey of the role that copulas play in the study of concordance and measures of association based on concordance.

2. Concordance

We begin with a definition of concordance [Hoeffding (1947)]: Two observations (x_1, y_1) and (x_2, y_2) of a pair (X, Y) of continuous random variables are *concordant* if both values of one pair are greater than the corresponding values of the other pair, that is if $x_1 < x_2,\ y_1 < y_2$ or $x_1 > x_2,\ y_1 > y_2$; and they are *discordant* if for one pair one value is greater and the other smaller than for the other pair, that is if $x_1 < x_2,\ y_1 > y_2$ or $x_1 > x_2,\ y_1 < y_2$.

The sample version of the measure of association known as Kendall's tau is defined in terms of concordance as follows [Kruskal (1958)]: Let $\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}$ denote a random sample of n observations from a vector (X, Y) of continuous random variables. There are $\binom{n}{2}$ distinct pairs (x_i, y_i) and (x_j, y_j) of observations in the sample, and each pair is either concordant or discordant. Kendall's tau for the sample is defined as

$$t = \frac{(\text{number of concordant pairs}) - (\text{number of discordant pairs})}{\text{total number of pairs}}.$$

Equivalently, t is the probability of concordance minus the probability of discordance for a pair of observations (x_i, y_i) and (x_j, y_j) chosen randomly from the sample. The population version of Kendall's tau for a vector (X, Y) of continuous random variables with joint distribution function H is defined similarly. Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed random vectors, each with joint distribution function H. Then the population version of Kendall's tau is defined as the probability of concordance minus the probability of discordance:

$$\tau = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0] \tag{2.1}$$

(we shall use Latin letters for sample statistics and Greek letters for the corresponding population parameters).

In order to demonstrate the role that copulas play in concordance and measures of association such as Kendall's tau, we define a "concordance function" Q as the difference of the probabilities of concordance and discordance between two vectors (X_1, Y_1) and (X_2, Y_2) of continuous random variables with (possibly) different joint distributions H_1 and H_2, but with common margins F and G. We then show that this function depends on the distributions of (X_1, Y_1) and (X_2, Y_2) only through their copulas.

Theorem 2.1. *Let (X_1, Y_1) and (X_2, Y_2) be independent vectors of continuous random variables with joint distribution functions H_1 and H_2, respectively, with common margins F (of X_1 and X_2) and G (of Y_1 and Y_2). Let C_1 and C_2 denote the copulas of (X_1, Y_1) and (X_2, Y_2), respectively, so that $H_1(x, y) = C_1(F(x), G(y))$ and $H_2(x, y) = C_2(F(x), G(y))$.*

Let Q denote the difference between the probabilities of concordance and discordance of (X_1, Y_1) and (X_2, Y_2), i.e., let

$$Q = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]. \quad (2.2)$$

Then

$$Q = Q(C_1, C_2) = 4 \iint_{\mathbf{I}^2} C_2(u,v) dC_1(u,v) - 1. \quad (2.3)$$

The proof is straightforward, see Nelsen (1999) for details.

Some of the useful properties of Q are summarized in the following corollary.

Corollary 2.2. *Let C_1, C_2, and Q be as given in Theorem 2.1. Then*

1. *Q is symmetric in its arguments: $Q(C_1, C_2) = Q(C_2, C_1)$.*
2. *Q is nondecreasing in each argument: if $C_1(u,v) \le C_1'(u,v)$ and $C_2(u,v) \le C_2'(u,v)$ for all (u,v) in $\mathbf{I}^2$, then $Q(C_1, C_2) \le Q(C_1', C_1')$.*

In part 2 of the above corollary, we say that C is *less concordant* than C' (and write $C \preceq C'$) whenever $C(u,v) \le C'(u,v)$ for all (u,v) in $\mathbf{I}^2$. For a discussion of concordance (and other) orders, see Kimeldorf and Sampson (1987, 1989).

The function Q is easily evaluated for pairs of the basic copulas M, W (the copulas of the Fréchet-Hoeffding upper and lower bounds) and Π (the copula of independent random variables):

$$Q(M,M) = 1, \; Q(W,W) = -1, \text{ and } Q(M,W) = 0;$$
$$Q(M,\Pi) = 1/3, \; Q(W,\Pi) = -1/3, \text{ and } Q(\Pi,\Pi) = 0.$$

If C is an arbitrary copula, then $Q(C,C) \in [-1,1]$ (since Q is the difference of two probabilities), and as a consequence of Corollary 2.2 and the values of Q displayed above, it also follows that

$$Q(C,M) \in [0,1], \; Q(C,W) \in [-1,0], \text{ and } Q(C,\Pi) \in [-1/3, 1/3]. \quad (2.4)$$

3. Measures of concordance

As an immediate consequence of (2.1), (2.2), (2.3), the population version of Kendall's tau has a succinct expression in terms of the concordance function Q:

Theorem 3.1. *Let X and Y be continuous random variables whose copula is C. Then the population version of Kendall's tau for X and Y (which we denote by either $\tau_{X,Y}$ or τ_C) is given by*

$$\tau_{X,Y} = \tau_C = Q(C,C) = 4 \iint_{\mathbf{I}^2} C(u,v) dC(u,v) - 1. \quad (3.1)$$

For computational purposes, there are alternate expressions for τ_C. The integral which appears in (3.1) can be interpreted as the expected value of the function $C(U,V)$ of uniform (0,1) random variables U and V whose joint distribution function is C, i.e.,

$$\tau_C = 4E(C(U,V)) - 1 = 4\int_0^1 t\,dK_C(t) - 1 = 3 - 4\int_0^1 K_C(t)dt$$

where K_C denotes the distribution function of the random variable $C(U,V)$. For example, if C is an Archimedean copula whose additive generator is ϕ, then $K_C(t) = t - (\phi(t)/\phi'(t^+))$ [Genest and MacKay (1986a,b)] and

$$\tau_C = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}dt.$$

As a second example, let C be the copula of random variables with a Bertino (1977) distribution, with diagonal section $\delta(t) = C(t,t)$. Then $K_C(t) = 2\delta^{(-1)}(t) - t$ where $\delta^{(-1)}$ denotes the right-continuous quasi-inverse of δ, i.e., $\delta^{(-1)}(t) = \sup\{s|\delta(s) \le t\}$; and

$$\tau_C = 5 - 8\int_0^1 \delta^{(-1)}(t)dt = 8\int_0^1 \delta(u)du - 3.$$

If C is singular or if C has a singular component, the form for τ_C given by (3.1) is not amenable to computation. For many such copulas, the expression

$$\tau_C = 1 - 4\iint_{\mathbf{I}^2} \frac{\partial}{\partial u}C(u,v)\frac{\partial}{\partial v}C(u,v)dudv \tag{3.2}$$

is more tractable. The equivalence of (3.1) and (3.2) is a consequence of the following theorem [Taylor (1999)]:

Theorem 3.2. *Let C_1 and C_2 be copulas. Then*

$$\iint_{\mathbf{I}^2} C_1(u,v)dC_2(u,v) = \frac{1}{2} - \iint_{\mathbf{I}^2} \frac{\partial}{\partial u}C_1(u,v)\frac{\partial}{\partial v}C_2(u,v)dudv.$$

For example, let $C_{\alpha,\beta}(0 \le \alpha, \beta \le 1)$ be a member of the generalized Cuadras-Augé (1981) family, the survival copulas for the Marshall-Olkin (1967) family of bivariate exponential distributions: $C_{\alpha,\beta}(u,v) = \min(u^{1-\alpha}v, uv^{1-\beta})$. Although there is a singular component on the curve $u^\alpha = v^\beta$ when $\alpha\beta > 0$, the partial derivatives of $C_{\alpha,\beta}$ are easily evaluated, and

$$\tau_{C_{\alpha,\beta}} = \alpha\beta/(\alpha - \alpha\beta + \beta).$$

As with Kendall's tau, the population version of Spearman's rho is based on concordance and discordance. To obtain the population version of this measure [Kruskal (1958)], we let (X_1,Y_1), (X_2,Y_2) and (X_3,Y_3) be three independent random vectors of continuous random variables with a common joint distribution function H (whose margins are again F and G) and copula C. The population version of

Spearman's rho is defined to be proportional to the probability of concordance minus the probability of discordance for the two vectors (X_1, Y_1) and (X_2, Y_3)—i.e., a pair of vectors with the same margins, but one vector has distribution function H, while the components of the other are independent:

$$\rho = 3(P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0]) \quad (3.3)$$

(the pair (X_3, Y_2) could be used equally as well). Note that while the joint distibution function of (X_1, Y_1) is $H(x, y)$, the joint distribution function of (X_2, Y_3) is $F(x)G(y)$ (since X_2 and Y_3 are independent). Thus the copula of X_2 and Y_3 is Π, and using Theorem 2.1, we immediately have

Theorem 3.3. *Let X and Y be continuous random variables whose copula is C. Then the population version of Spearman's rho for X and Y (which we denote be either $\rho_{X,Y}$ or ρ_C) is given by*

$$\rho_{X,Y} = \rho_C = 3Q(C, \Pi). \quad (3.4)$$

The coefficient "3" which appears in (3.3) and (3.4) is a normalization constant, since as noted in (2.4), $Q(C, \Pi) \in [-1/3, 1/3]$. Combining (3.4) with (2.3) yields

$$\begin{aligned} \rho_C &= 12 \iint_{\mathbf{I}^2} uv\, dC(u, v) - 3 = 12 \iint_{\mathbf{I}^2} C(u, v)\, du dv - 3 \\ &= 12 \iint_{\mathbf{I}^2} [C(u, v) - uv]\, du dv. \end{aligned} \quad (3.5)$$

Another measure based on concordance, less well-known than Kendall's tau and Spearman's rho, is Gini's "coefficient of cograduation". Noting that Spearman's rho can be written as

$$\rho_C = 3 \iint_{\mathbf{I}^2} ([u + v - 1]^2 - [u - v]^2)\, dC(u, v),$$

Gini considered a measure γ based on absolute values rather than squares:

$$\gamma = 2 \iint_{\mathbf{I}^2} (|u + v - 1| - |u - v|)\, dC(u, v).$$

The relationship between γ and the concordance function Q is given by the following theorem, whose proof can be found in Nelsen (1998, 1999).

Theorem 3.4. *Let X and Y be continuous random variables whose copula is C. Then the population version of Gini's measure of assocation for X and Y (which we denote by $\gamma_{X,Y}$ or γ_C) is given by*

$$\gamma_{X,Y} = \gamma_C = Q(C, M) + Q(C, W).$$

In a sense, Spearman's $\rho_C = 3Q(C, \Pi)$ measures a concordance relationship between the distribution of X and Y as represented by their copula C, and independence as represented by the copula Π. On the other hand, Gini's $\gamma_C = Q(C, M) + Q(C, W)$ measures a concordance relationship between C and monotone dependence, as represented by the copulas M and W.

The measures of association in this section are often called "measures of concordance", not because they are based on the concordance function Q, but because they satisfy a set of axioms proposed by Scarsini (1984):

Definition 3.5. A numeric measure κ of association between two continuous random variables X and Y whose copula is C is a *measure of concordance* if it satisfies the following properties (again we write $\kappa_{X,Y}$ or κ_C when convenient):

1. κ is defined for every pair X, Y of continuous random variables;
2. $-1 \leq \kappa_{X,Y} \leq 1$, $\kappa_{X,X} = 1$, and $\kappa_{X,-X} = -1$;
3. $\kappa_{X,Y} = \kappa_{Y,X}$;
4. if X and Y are independent, then $\kappa_{X,Y} = \kappa_{\Pi} = 0$;
5. $\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}$;
6. if C_1 and C_2 are copulas such that $C_1 \preceq C_2$, then $\kappa_{C_1} \leq \kappa_{C_2}$;
7. if $\{(X_n, Y_n)\}$ is a sequence of continuous random variables with copulas C_n, and if $\{C_n\}$ converges pointwise to C, then $\lim_{n\to\infty} \kappa_{C_n} = \kappa_C$.

The following consequences of Definition 3.5 are readily established:

8. if Y is almost surely an increasing function of X, then $\kappa_{X,Y} = \kappa_M = 1$; and if Y is almost surely a decreasing function of X, then $\kappa_{X,Y} = \kappa_W = -1$;
9. if α and β are almost surely strictly monotone functions on the range of X and the range of Y, respectively, then $\kappa_{\alpha(X),\beta(Y)} = \kappa_{X,Y}$.

Theorem 3.6. *The population versions of Kendall's tau, Spearman's rho, and Gini's gamma are measures of concordance.*

We note that not all measures of association derived from the concordance function Q are measures of concordance. For example, the population version of Spearman's footrule, $\varphi = [3Q(C, M) - 1]/2$, does not satisfy parts 2 and 5 of Definition 3.5.

One defect of a measure of concordance is the fact that the converse of part 4 of Definition 3.5 does not hold—$\kappa_{X,Y}$ may be 0 even when X and Y are dependent. Schweizer and Wolff (1981) proposed the measure

$$\sigma_{X,Y} = 12 \iint_{\mathbf{I}^2} |C(u, v) - uv| dudv, \tag{3.6}$$

which is similar to the third expression for Spearman's rho in (3.5). While $\sigma_{X,Y}$ is not a measure of concordance, $\sigma_{X,Y} = 0$ if and only if X and Y are independent. Similar measures of "monotone dependence" related to Kendall's tau and Gini's gamma are

$$\tau^*_{X,Y} = 6 \iint_{\mathbf{I}^2} |C(u,v) - uv| dC(u,v) \text{ and} \tag{3.7}$$

$$\gamma^*_{X,Y} = 4 \iint_{\mathbf{I}^2} |C(u,v) - uv| d[M(u,v) + W(u,v)]. \tag{3.8}$$

We conclude this section with one additional measure of association based on concordance. Suppose in the expression (2.2) for Q, we use a random vector and a fixed point, rather than two random vectors. Blomqvist (1950) proposed and studied such a measure, often called the "medial correlation coefficient", which we denote as β:

$$\beta = \beta_{X,Y} = P[(X - \tilde{x})(Y - \tilde{y}) > 0] - P[(X - \tilde{x})(Y - \tilde{y}) < 0]$$

where $\tilde{x}$ and $\tilde{y}$ are medians of X and Y, respectively. While β does not have the form given by (2.3), there is a simple expression for β when X and Y are continuous with copula C: $\beta = \beta_C = 4C(1/2, 1/2) - 1$. Like τ, ρ and γ, β satisfies the properties in Definition 3.5 for a measure of concordance.

For example, if C_θ is a member of the Frank (1979) family of copulas

$$C_\theta(u,v) = -\ln[1 + (e^{-\theta u} - 1)(e^{-\theta v} - 1)/(e^{-\theta} - 1)]/\theta, \theta \in \mathbf{R},$$

then τ_{C_θ} and ρ_{C_θ} involve Debye functions of θ, whereas

$$\beta_{C_\theta} = (4/\theta) \ln \cosh(\theta/4).$$

4. Multivariate concordance and measures

To simplify the presentation, we introduce some notation. Let $\mathbf{X} = (X_1, X_2, \cdots, X_n)$, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, and let $\mathbf{X} > \mathbf{x}$ denote the component-wise inequality. If $\mathbf{U}$ is a vector of uniform $(0,1)$ random variables whose distribution function is the n-copula C, then $\overline{C}$ denotes the survival function, $\overline{C}(\mathbf{u}) = P[\mathbf{U} > \mathbf{u}]$. Lastly, M^n denotes the Fréchet-Hoeffding upper bound n-copula $\min(u_1, u_2, \cdots, u_n)$, and Π^n the n-copula $u_1 u_2 \cdots u_n$ of independent random variables.

We now generalize the notion of concordance and the concordance function Q from Section 2. Suppose we have two observations $\mathbf{x}$ and $\mathbf{y}$ of a vector $\mathbf{X}$ of continuous random variables. Concordance generalizes: $\mathbf{x}$ and $\mathbf{y}$ are concordant if for all $i \neq j$, (x_i, x_j) and (y_i, y_j) are concordant. However, discordance does not generalize to dimensions $n \geq 3$: if (x_1, x_2) and (y_1, y_2) are discordant and (x_2, x_3) and (y_2, y_3) are discordant, then (x_1, x_3) and (y_1, y_3) must be concordant. Consequently, we consider the probability of concordance alone, rather than the difference of the probabilities of concordance and discordance. For a complete discussion of multivariate concordance, see Joe (1990).

The next theorem, whose proof is analogous to that of Theorem 2.1, presents the probability of concordance in terms of n-copulas.

Theorem 4.1. *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be independent vectors of continuous random variables with common univariate margins and n-copulas C_1 and C_2, respectively, and let Q'_n denote the probability of concordance between $\mathbf{X}_1$ and $\mathbf{X}_2$:*

$$Q'_n = P[\mathbf{X}_1 > \mathbf{X}_2] + P[\mathbf{X}_1 < \mathbf{X}_2].$$

Then

$$\begin{aligned} Q'_n = Q'_n(C_1, C_2) &= \textstyle\int_{\mathbf{I}^n} C_2(\mathbf{u}) dC_1(\mathbf{u}) + \int_{\mathbf{I}^n} C_1(\mathbf{u}) dC_2(\mathbf{u}), \\ &= \textstyle\int_{\mathbf{I}^n} [C_2(\mathbf{u}) + \overline{C}_2(\mathbf{u})] dC_1(\mathbf{u}). \end{aligned}$$

Like the function Q in (2.3), Q'_n is symmetric in its arguments, nondecreasing with respect to the multivariate concordance ordering ($C \preceq C'$ if $C(\mathbf{u}) \leq C'(\mathbf{u})$ and $\overline{C}(\mathbf{u}) \leq \overline{C'}(\mathbf{u})$ for all $\mathbf{u}$ in $\mathbf{I}^n$), and is easily evaluated for pairs of the n-copulas M^n and Π^n (recall that the Fréchet-Hoeffding lower bound $W^n = \max(u_1 + u_2 + \cdots + u_n - n + 1, 0)$ is not an n-copula for $n \geq 3$): $Q'_n(M^n, M^n) = 1$, $Q'_n(M^n, \Pi^n) = 2/(n+1)$, and $Q'_n(\Pi^n, \Pi^n) = 1/2^{n-1}$. We now define a multivariate analog of Q (which we denote Q_n) for continuous random vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ whose copulas are C_1 and C_2 as a linear function of Q'_n:

$$Q_n(C_1, C_2) = \frac{1}{2^{n-1} - 1}[2^{n-1} Q'_n(C_1, C_2) - 1].$$

Consequently,

$$\begin{gathered} Q_n(M^n, M^n) = 1,\ Q_n(\Pi^n, \Pi^n) = 0, \text{ and} \\ Q_n(M^n, \Pi^n) = \frac{2^n - (n+1)}{(n+1)(2^{n-1} - 1)}. \end{gathered}$$

Multivariate analogues of Spearman's rho and Kendall's tau now follow:

Definition 4.2. Let $\mathbf{X}$ be continuous random vector whose n-copula is C. Population versions $\tau_{n,C}$ of Kendall's tau and $\rho_{n,C}$ of Spearman's rho are given by

$$\tau_{n,C} = Q_n(C, C) \text{ and } \rho_{n,C} = \frac{(n+1)(2^{n-1} - 1)}{2^n - (n+1)} Q_n(C, \Pi^n). \tag{4.1}$$

Explicitly, we have

$$\tau_{n,C} = \frac{1}{2^{n-1}-1}\left[2^n \int_{\mathbf{I}^n} C(\mathbf{u})dC(\mathbf{u}) - 1\right] \text{ and}$$

$$\rho_{n,C} = \frac{n+1}{2^n-(n+1)}\left[2^{n-1}(\int_{\mathbf{I}^n} C(\mathbf{u})d\Pi^n(\mathbf{u}) + \int_{\mathbf{I}^n} \Pi^n(\mathbf{u})dC(\mathbf{u})) - 1\right].$$

The coefficient $\tau_{n,C}$ in (4.1) is one of a family of generalizations of Kendall's tau discussed by Joe (1990), while $\rho_{n,C}$ is the mean of two generalizations of Spearman's rho discussed by Joe (1990). These coefficients also appear in Nelsen (1996) as measures of average multivariate total positivity of order two, and of average upper and lower orthant dependence. An analogous argument yields a multivariate version of Blomqvist's coefficient: $\beta_{n,C} = [2^n C(\mathbf{1/2}) - 1]/(2^{n-1} - 1)$.

We conclude with several open problems: 1) Develop a multivariate version of Gini's coefficient; 2) formulate a definition for measures of multivariate concordance comparable to Definition 3.5; and 3) develop multivariate versions of the measures of monotone dependence given in (3.6), (3.7), and (3.8).

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