Available online at www.sciencedirect.com

ScienceDirect

Insurance: Mathematics and Economics 42 (2008) 473–483

INSURANCE MATHEMATICS & ECONOMICS

www.elsevier.com/locate/ime

On the construction of copulas and quasi-copulas with given diagonal sections[☆]

Roger B. Nelsen[a], José Juan Quesada-Molina[b,*], José Antonio Rodríguez-Lallena[c], Manuel Úbeda-Flores[c]

[a] Department of Mathematical Sciences, Lewis & Clark College, 0615 S.W. Palatine Hill Road, Portland, OR 97219, USA
[b] Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain
[c] Departamento de Estadística y Matemática Aplicada, Universidad de Almería, 04120 Almería, Spain

Received September 2006; received in revised form November 2006; accepted 26 November 2006

Abstract

We study a method, which we call a copula (or quasi-copula) *diagonal splice*, for creating new functions by joining portions of two copulas (or quasi-copulas) with a common diagonal section. The diagonal splice of two quasi-copulas is always a quasi-copula, and we find a necessary and sufficient condition for the diagonal splice of two copulas to be a copula. Applications of this method include the construction of absolutely continuous asymmetric copulas with a prescribed diagonal section, and determining the best-possible upper bound on the set of copulas with a particular type of diagonal section. Several examples illustrate our results.

© 2006 Elsevier B.V. All rights reserved.

MSC: 60E15; 62E10; 62H20

Keywords: Bounds; Copulas; Diagonal sections; Distribution functions; Quasi-copulas

---

1. Introduction

The construction of distributions with given marginals has been a problem of interest to statisticians for many years. Today, in view of *Sklar's theorem* (Sklar, 1959), this problem can be reduced to the construction of a copula. Nelsen (2006) summarizes different methods of constructing copulas. Copulas have been used, among many other purposes, to find best-possible bounds on sets of distribution functions: see, for instance, Nelsen et al. (2004), Nelsen and Úbeda-Flores (2005) and Rodríguez-Lallena and Úbeda-Flores (2004). In this paper we will only deal with bivariate copulas and quasi-copulas. Thus, in the sequel we will usually omit the word "bivariate".

A *copula* is a function $C\colon [0,1]^2 \to [0,1]$ which satisfies:

(C1) the *boundary conditions* $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for all t in $[0,1]$; and
(C2) the 2-*increasing property*, i.e., $V_C([u_1,u_2]\times[v_1,v_2]) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$ for all u_1,u_2,v_1,v_2 in $[0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

The rectangle $[u_1,u_2]\times[v_1,v_2]$ is called a 2-*box*. Equivalently, a copula is the restriction to $[0,1]^2$ of a continuous bivariate distribution function whose margins are uniform on $[0,1]$.

The importance of copulas as a tool for statistical analysis and modelling stems largely from the observation that the joint distribution H of the random pair (X,Y) with respective margins F and G can be expressed by $H(x,y) = C(F(x),G(y))$ for every $(x,y) \in [-\infty,\infty]^2$, where C is a copula that is uniquely determined on Range F × Range G (Sklar's theorem).

If the random variables are *exchangeable*, i.e., if the random vectors (X,Y) and (Y,X) are identically distributed, then the copula C of (X,Y) is *symmetric*, i.e., $C(u,v) = C(v,u)$ for all $(u,v) \in [0,1]^2$. Observe that, given a copula C, the

---

[☆] This work was partially supported by the Ministerio de Ciencia y Tecnología (Spain) and FEDER, under research project BFM2003–06522, and also by the Consejería de Educación y Ciencia of the Junta de Andalucía (Spain).

[*] Corresponding author. Tel.: +34 958 249022; fax: +34 958 249513.
E-mail addresses: nelsen@lclark.edu (R.B. Nelsen), jquesada@ugr.es (J.J. Quesada-Molina), jarodrig@ual.es (J.A. Rodríguez-Lallena), mubeda@ual.es (M. Úbeda-Flores).

0167-6687/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2006.11.011

function C^t defined on $[0,1]^2$ by $C^t(u,v) = C(v,u)$ is also a copula. C^t will be called the *transpose* of the copula C. Note that a copula C is symmetric if and only if $C^t = C$. Many, perhaps most, of the copulas encountered in the literature are symmetric. However, exchangeability of random variables is rather uncommon in real life. In this paper we deal with the common situation in which the copula C is *asymmetric*, i.e., $C^t \neq C$. The definitions of symmetric and asymmetric copulas, and the transpose of a copula can be extended to any function defined on $[0,1]^2$.

The concept of a quasi-copula is a more general notion than that of a copula. It was introduced by Alsina et al. (1993) – see Nelsen et al. (1996) for the multivariate case – in order to characterize operations on distribution functions that can or cannot be derived from operations on random variables defined on the same probability space. A *quasi-copula* – see Genest et al. (1999) for more details – is a function $Q\colon [0,1]^2 \to [0,1]$ which satisfies condition (C1), but in place of (C2), the weaker conditions:

(Q1) Q is nondecreasing in each variable; and
(Q2) the *Lipschitz condition*

$$|Q(u_1,v_1) - Q(u_2,v_2)| \le |u_1 - u_2| + |v_1 - v_2| \quad \text{for all } (u_1,v_1), (u_2,v_2) \text{ in } [0,1]^2.$$

While every copula is a quasi-copula, there exist *proper* quasi-copulas, i.e., quasi-copulas which are not copulas.

In the sequel, for any two functions A and B defined on a common domain $\mathcal{D}$, $A \le B$ will denote $A(\mathbf{x}) \le B(\mathbf{x})$ for every $\mathbf{x} \in \mathcal{D}$.

Let W, Π and M denote the copulas defined by $W(u,v) = \max(u+v-1,0)$, $\Pi(u,v) = uv$ and $M(u,v) = \min(u,v)$ for every $(u,v) \in [0,1]^2$. W and M are known as the *Fréchet–Hoeffding bounds* for copulas and quasi-copulas, since $W \le Q \le M$ for any quasi-copula (in particular, for any copula) Q; and Π is known as the *independence copula*: see Nelsen (2006) for more details.

The *diagonal section* δ_C of a copula C (and similarly for a quasi-copula) is the function defined by $\delta_C(t) = C(t,t)$ for every $t \in [0,1]$. On the other hand, a *diagonal* is a function $\delta\colon [0,1] \to \mathbb{R}$ which satisfies the following conditions:

(i) $\delta(1) = 1$,
(ii) $\delta(t) \le t$ for every $t \in [0,1]$, and
(iii) $0 \le \delta(t') - \delta(t) \le 2(t'-t)$ for every $t, t' \in [0,1]$ such that $t \le t'$.

The diagonal section of any quasi-copula (or copula) is a diagonal; and for any diagonal δ, there exist copulas (and quasi-copulas) whose diagonal section is δ (Fredricks and Nelsen, 1997; Nelsen and Fredricks, 1997; Nelsen et al., 2004).

The diagonal section of a copula C has several probabilistic interpretations (Nelsen, 2006; Nelsen et al., 2001, 2004); for instance, δ_C is the restriction to $[0,1]$ of the distribution function of $\max(U,V)$ whenever (U,V) is a random pair distributed as C. More generally, if (X,Y) is distributed according to H, with respective margins F and G and copula C, and $x_t, y_t \in \mathbb{R}$ are respective $100t$-th percentiles for every $t \in (0,1)$, then $\delta_C(t) = \Pr[X \le x_t, Y \le y_t]$ for every $t \in (0,1)$. Furthermore, δ_C can be used to study the *tail dependence* of the random pair (X,Y) (Nelsen, 2006): the *upper* and *lower tail dependence parameters* λ_U and λ_L, which are defined as $\lambda_U = \lim_{t\to 1^-} \Pr[Y > y_t | X > x_t]$ and $\lambda_L = \lim_{t\to 0^+} \Pr[Y \le y_t | X \le x_t]$ (if the limits exist), can be computed as follows: $\lambda_U = 2 - \delta'_C(1^-)$ and $\lambda_L = \delta'_C(0^+)$.

Copulas are used to build models for dependence between risks in financial and actuarial risk management, especially dependence between extreme events (Bäuerle and Müller, 1998; Denuit et al., 2005; Frees and Valdez, 1998; Klugman and Parsa, 1999). Tail dependence has been shown to be useful for describing this dependence, in particular in volatile and bear markets (Ané and Kharoubi, 2003; Malevergne and Sornette, 2006), and in contagion and stress testing concepts (Abdous et al., 2005). For other applications, see Embrechts et al. (2002), Frahm et al. (2005), Frahm et al. (2003) and Schmidt (2002). Since, as we have just observed, tail dependence is a property of the diagonal section of the copula in the model, creating copulas with given diagonal section but with a variety of dependence structures has applications in insurance and finance.

In Section 2, after some preliminary concepts and results, we introduce a method for constructing copulas and quasi-copulas with a given diagonal section. An alternative approach to that method can be found in an unpublished paper of Durante et al. (2007). We show that, in particular, such a method can be used to construct absolutely continuous asymmetric copulas.

In Section 3, we first review best-possible bounds for the sets of all copulas or quasi-copulas with a common diagonal section (see Úbeda-Flores (2001) for a preliminary study). In this study, the only problematic case is the obtaining of the best-possible upper bound for the set of copulas with a given diagonal section. We introduce the concept of a *simple* diagonal and show that many of the most commonly used copulas have simple diagonal sections. We also show that an important subclass of such diagonals is the convex diagonals. We find an elementary way to construct asymmetric copulas with simple diagonal sections and, as an application, we obtain the best-possible upper bound for the set of copulas with a given simple diagonal.

2. Construction of copulas with a given diagonal section

We begin this section with some notation which will be useful in the sequel.

Consider the triangles T_U and T_L in $[0,1]^2$, defined by $T_U = \{(u,v) \in [0,1]^2 : u \le v\}$ and $T_L = \{(u,v) \in [0,1]^2 : u \ge v\}$; and their intersection $D = T_L \cap T_U$, which is the diagonal of the unit square given by $D = \{(u,u) \in [0,1]^2 : u \in [0,1]\}$. For any $u, v \in [0,1]$, $[\{u,v\}]$ denotes the interval $[\min(u,v), \max(u,v)]$.

Let δ be any diagonal. Then, $\mathbf{C}_\delta$ (respectively, $\mathbf{Q}_\delta$) denotes the set of all copulas (respectively, quasi-copulas) whose diagonal section is δ. It is known (Bertino, 1977; Fredricks and Nelsen, 2002; Nelsen and Fredricks, 1997; Nelsen et al., 2004) that $\mathbf{C}_\delta$ and $\mathbf{Q}_\delta$ are nonempty sets for any diagonal δ. Of course, $\mathbf{C}_\delta \subset \mathbf{Q}_\delta$, and this inclusion is usually strict. An exception occurs when $\delta = \delta_M$: in this case $\mathbf{C}_\delta = \mathbf{Q}_\delta = \{M\}$. It is an

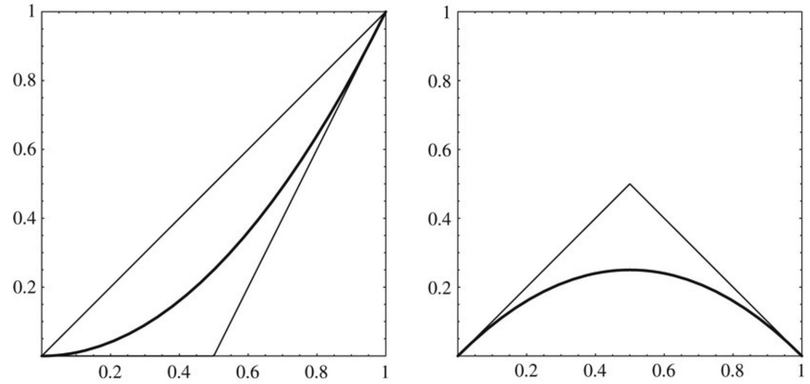

Fig. 1. The graphs of δ and $\hat{\delta}$ for the independence copula Π, and the triangles wherein those graphs lie.

open problem to determine whether such an inclusion is strict for all other diagonals. Another open problem is to determine whether there exist other diagonals δ such that $\mathbf{C}_\delta$ is a singleton.

Associated with δ is the function $\hat{\delta}$ defined by

$$\hat{\delta}(t) = t - \delta(t), \quad t \in [0, 1]. \tag{1}$$

Let U and V be uniform [0, 1] random variables, and C the copula of the pair (U, V). We now provide a probabilistic interpretation of the function $\hat{\delta}_C$. It is easy to check that $\Pr[\min(U, V) \le t] = t + \hat{\delta}_C(t)$ and $\Pr[\max(U, V) \le t] = t - \hat{\delta}_C(t)$, whence

$$\hat{\delta}_C(t) = \frac{1}{2}\Pr[\min(U, V) \le t < \max(U, V)]. \tag{2}$$

The following result provides properties of the function $\hat{\delta}$ which are needed later.

Theorem 1. *Let δ be a diagonal, and let $\hat{\delta}$ be the function defined by (1). Then, the following properties hold:*

(i) $\hat{\delta}(0) = \hat{\delta}(1) = 0$.
(ii) $|\hat{\delta}(t') - \hat{\delta}(t)| \le |t' - t|$ *for every* $t, t' \in [0, 1]$.
(iii) $0 \le \hat{\delta}(t) \le \min(t, 1 - t)$ *for all* $t \in [0, 1]$.

Proof. Property (i) is trivial. To prove property (ii), suppose, without loss of generality, $t < t'$. Then it suffices to show that, for every $t, t' \in [0, 1]$ such that $t < t'$, we have $-(t' - t) \le \hat{\delta}(t') - \hat{\delta}(t) = t' - t - (\delta(t') - \delta(t)) \le t' - t$; but these inequalities are immediately equivalent to condition (iii) of the definition of a diagonal. Taking $t' = 1$ in this last condition, we have that $\max(2t - 1, 0) \le \delta(t) = t - \hat{\delta}(t)$ for all $t \in [0, 1]$, whence $\hat{\delta}(t) \le t - \max(2t - 1, 0) = \min(1 - t, t)$ for all $t \in [0, 1]$. Finally, the first inequality of property (iii) follows from condition (ii) of the definition of a diagonal. ■

The conditions in the definition of a diagonal yield that the graph of δ lies in the triangle with vertices (0, 0), (1/2, 0) and (1, 1), and its slope at each point takes values in the interval [0, 2]. On the other hand, as a consequence of Theorem 1, the graph of $\hat{\delta}$ is in the triangle with vertices (0, 0), (1/2, 1/2) and (1, 0), and its slope at each point takes values in the interval [−1, 1]. In Fig. 1 we show the graphs of a diagonal δ and the respective function $\hat{\delta}$, and also the triangles where such graphs are included.

For any diagonal δ, h_δ and l_δ will denote the functions defined by

$$l_\delta(u, v) = \min(\hat{\delta}(t) \mid t \in [\{u, v\}]), \quad (u, v) \in [0, 1]^2,$$
$$h_\delta(u, v) = \max(\hat{\delta}(t) \mid t \in [\{u, v\}]), \quad (u, v) \in [0, 1]^2. \tag{3}$$

Observe that l_δ and h_δ are symmetric.

In the following theorem we review three different ways to construct copulas and quasi-copulas with given diagonal sections, and some of their known properties (Fredricks and Nelsen, 1997, 2002; Nelsen and Fredricks, 1997; Nelsen et al., 2004; Úbeda-Flores, 2001).

Theorem 2. *Let δ be a diagonal, and B_δ, K_δ and A_δ the functions defined by*

$$B_\delta(u, v) = \min(u, v) - l_\delta(u, v), \tag{4}$$

$$K_\delta(u, v) = \min\left(u, v, \frac{\delta(u) + \delta(v)}{2}\right) \tag{5}$$

and

$$A_\delta(u, v) = \min(u, v, \max(u, v) - h_\delta(u, v)), \tag{6}$$

for every $(u, v) \in [0, 1]^2$. Then the following statements hold:

(i) *B_δ and K_δ are copulas (i.e., $B_\delta, K_\delta \in \mathbf{C}_\delta$). Moreover, both are singular.*
(ii) *A_δ is a quasi-copula (i.e., $A_\delta \in \mathbf{Q}_\delta$).*
(iii) *B_δ, K_δ and A_δ are symmetric, and satisfy $B_\delta \le K_\delta \le A_\delta$.*
(iv) *If C is in $\mathbf{C}_\delta$ and symmetric, then $C \le K_\delta$.*
(v) *A_δ is a copula if and only if $A_\delta = K_\delta$; and a necessary and sufficient condition for this equality is the following: the graph of δ is piecewise linear, and each segment of the graph satisfies* (a) *its slope is* 0,1 *or* 2 *and* (b) *at least one of its endpoints is on the line $v = u$.*

We conjecture that A_δ is a singular quasi-copula – i.e., $(\partial^2 A_\delta/\partial u \partial v)(u, v) = 0$ almost everywhere in $[0, 1]^2$ – for every diagonal δ. Observe that, to prove this conjecture, it is sufficient to consider the case when A_δ is a proper quasi-copula: otherwise $A_\delta = K_\delta$, and K_δ is singular.

We now define an operation on the set of real functions defined on the square $[0, 1]^2$.

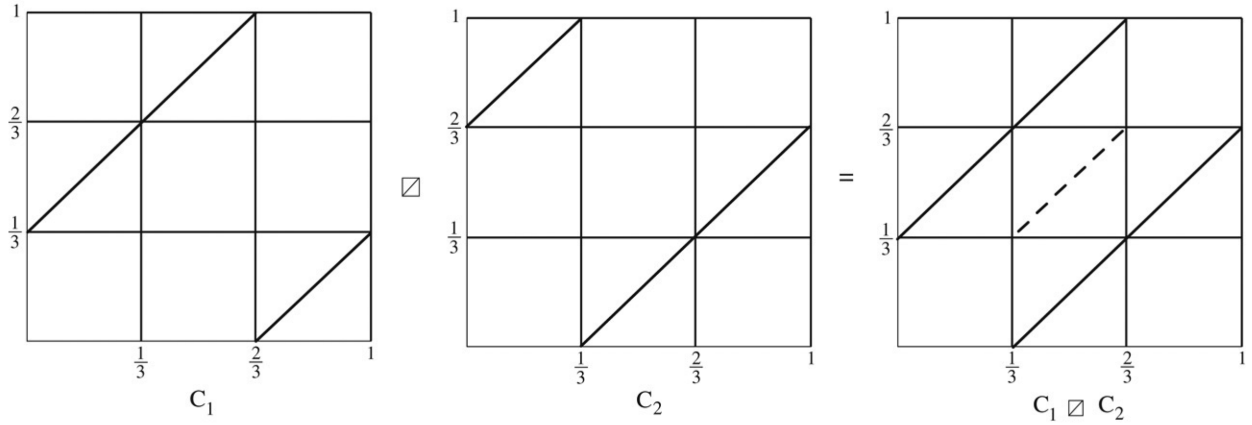

Fig. 2. An example where the splice of two copulas C_1 and C_2 is a proper quasi-copula.

Definition 3. Let f_1 and f_2 be two functions defined on the square $[0, 1]^2$. Then, the *diagonal splice* of f_1 and f_2 is the function $f_1 \boxslash f_2$ defined by

$$(f_1 \boxslash f_2)(u, v) = \begin{cases} f_1(u, v), & (u, v) \in T_U, \\ f_2(u, v), & (u, v) \in T_L \setminus D. \end{cases}$$

The diagonal splice $f_2 \boxslash f_1$ is defined similarly.

The diagonal splice operation was introduced in Durante et al. (2005) – under a different appearance – for *binary aggregation operators*, and applied to quasi-copulas. They also provided – without proof – the following result:

Theorem 4. *Let δ be a diagonal, and Q_1 and Q_2 two quasi-copulas in $\mathbf{Q}_\delta$. Then, the diagonal splice of Q_1 and Q_2 is also a quasi-copula in $\mathbf{Q}_\delta$.*

Proof. It is immediate that $Q_1 \boxslash Q_2$ satisfies the boundary conditions (C1). Let $u, u', v \in [0, 1]$ such that $u < u'$. We have to prove that $(Q_1 \boxslash Q_2)(u, v) \leq (Q_1 \boxslash Q_2)(u', v)$. If both (u, v) and (u', v) are in the triangle T_U, then such an inequality follows immediately; the same thing occurs when $(u, v), (u', v) \in T_L$; otherwise, if $(u, v) \in T_U \setminus D$ and $(u', v) \in T_L \setminus D$, then $(Q_1 \boxslash Q_2)(u, v) = Q_1(u, v) \leq Q_1(v, v) = \delta(v) = Q_2(v, v) \leq Q_2(u', v) = (Q_1 \boxslash Q_2)(u', v)$. Hence, $Q_1 \boxslash Q_2$ is nondecreasing in the first variable; analogously, it can be proved that $Q_1 \boxslash Q_2$ is nondecreasing in the second variable, whence $Q_1 \boxslash Q_2$ satisfies condition (Q1) of quasi-copulas. Recall that the Lipschitz condition (Q2) of quasi-copulas is equivalent to the corresponding Lipschitz conditions in each variable separately. Now, we prove that $Q_1 \boxslash Q_2$ satisfies such a condition for the first variable (the proof for the second variable is analogous). Let $u, u', v \in [0, 1]$, and suppose, without loss of generality, that $u < u'$. As for condition (Q1), we only need to prove the Lipschitz condition for the case $(u, v) \in T_U \setminus D$, $(u', v) \in T_L \setminus D$. In this case, $|(Q_1 \boxslash Q_2)(u, v) - (Q_1 \boxslash Q_2)(u', v)| = (Q_1 \boxslash Q_2)(u', v) - (Q_1 \boxslash Q_2)(u, v) = Q_2(u', v) - Q_1(u, v) = Q_2(u', v) - Q_2(v, v) + Q_1(v, v) - Q_1(u, v) \leq u' - v + v - u = u' - u = |u - u'|$, whence the proof follows. ■

The above result shows that "$\boxslash$" is a binary operation on $\mathbf{Q}_\delta$. Therefore, $(\mathbf{Q}_\delta, \boxslash)$ is a noncommutative semigroup in which every element is idempotent.

Theorem 4 cannot be restricted to either copulas or proper quasi-copulas, i.e., if Q_1 and Q_2 are copulas (respectively proper quasi-copulas) with a common diagonal section δ, then the diagonal splice of Q_1 and Q_2 can be a proper quasi-copula (respectively a copula), as the following examples show.

Example 5. Let C_1 and C_2 be the copulas whose probability mass is uniformly spread on two segments as shown in the first two graphs of Fig. 2, i.e., C_1 and $C_2 (= C_1^t)$ are the shuffles of Min (Mikusiński et al., 1992) given by: $C_1(u, v) = W(u, v)$ if $(u, v) \in ([0, 2/3] \times [0, 1/3]) \cup ([2/3, 1] \times [1/3, 1])$, and $C_1(u, v) = \max(\min(u, v - 1/3), \min(u - 2/3, v))$ otherwise; $C_2(u, v) = W(u, v)$ if $(u, v) \in ([0, 1/3] \times [0, 2/3]) \cup ([1/3, 1] \times [2/3, 1])$, and $C_2(u, v) = \max(\min(u, v - 2/3), \min(u - 1/3, v))$ otherwise. Observe that $\delta_{C_1}(t) = \delta_{C_2}(t) = \max(0, t - 1/3, 2t - 1)$. Then, it is easy to check that $(C_1 \boxslash C_2)(u, v) = W(u, v)$ if $(u, v) \in [0, 1/3]^2 \cup [2/3, 1]^2$, and $(C_1 \boxslash C_2)(u, v) = \max(\min(u, v - 2/3), \min(u - 2/3, v))$ otherwise; i.e., the diagonal splice $C_1 \boxslash C_2$ is the proper quasi-copula which spreads uniformly a mass of $-1/3$ on the segment joining the points $(1/3, 1/3)$ and $(2/3, 2/3)$, and a total mass of $4/3$ on the two other segments shown in the third graph of Fig. 2.

Example 6. Let Q_1 be the proper quasi-copula which spreads uniformly a mass of $1/4$ on the segment joining the points $(0, 3/4)$ and $(1/4, 1)$, a mass of $1/2$ on each one of the segments joining the points $(1/4, 1/4)$ and $(3/4, 3/4)$, and the points $(1/2, 0)$ and $(1, 1/2)$, and a mass of $-1/4$ on the segment joining the points $(1/2, 1/4)$ and $(3/4, 1/2)$. Let $Q_2 = Q_1^t$. Then, it is easy to check that the diagonal splice of Q_1 and Q_2 is a copula, specifically the shuffle of Min whose probability mass is uniformly spread on the segments joining the points $(0, 3/4)$ and $(1/4, 1)$, $(1/4, 1/4)$ and $(3/4, 3/4)$, and $(3/4, 0)$ and $(1, 1/4)$, as shown in Fig. 3.

With regard to the remaining cases, i.e., the diagonal splice of a copula and a proper quasi-copula (and vice versa), all the results are possible; see Theorem 31 and Example 33 below.

The following theorem provides a necessary and sufficient condition for the diagonal splice of two copulas with a common diagonal section to be a copula:

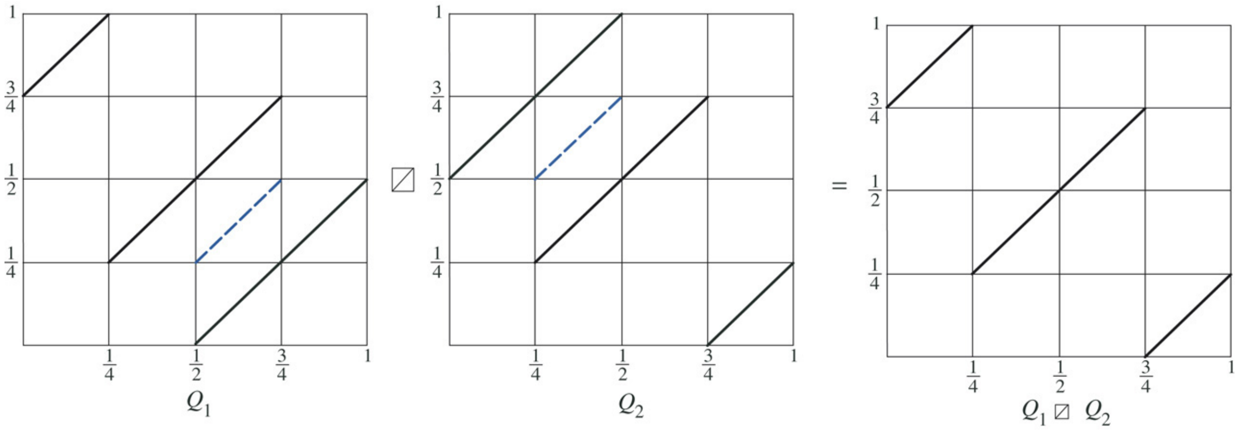

Fig. 3. An example where the splice of two proper quasi-copulas is a copula.

Theorem 7. *Let δ be a diagonal, and C_1 and C_2 two copulas in $\mathbf{C}_\delta$. Then, the diagonal splice $C_1 \boxslash C_2$ is a copula in $\mathbf{C}_\delta$ if, and only if, $C_1(u,v) + C_2(v,u) \le \delta(u) + \delta(v)$ for every $(u,v) \in T_U$; and similarly $C_2 \boxslash C_1$ is a copula in $\mathbf{C}_\delta$ if, and only if, $C_1(u,v)+C_2(v,u) \le \delta(u)+\delta(v)$ for every $(u,v) \in T_L$.*

Proof. We prove the first equivalence in the conclusion of this theorem; the second one can be proved analogously. The only nontrivial part of the proof of the first equivalence is the 2-increasingness of $C_1 \boxslash C_2$. From the definition of $C_1 \boxslash C_2$, it is immediate that $V_{C_1 \boxslash C_2}(J) \ge 0$ for any 2-box J included in either T_U or T_L. So we only need to prove that $V_{C_1 \boxslash C_2}(J) \ge 0$ for the remaining 2-boxes in $[0,1]^2$. It is clear that any of such 2-boxes can be decomposed as unions of a 2-box of the form $[u,v]^2$ with $(u,v) \in T_U \setminus D$, and (perhaps) others included in either T_U or T_L. Then, $C_1 \boxslash C_2$ is 2-increasing if and only if $V_{C_1 \boxslash C_2}([u,v]^2) \ge 0$ for every $(u,v) \in T_U$. Since $V_{C_1 \boxslash C_2}([u,v]^2) = \delta(v) - C_1(u,v) - C_2(v,u) + \delta(u)$, the conclusion follows. ■

Since the tail dependence parameters of a random pair only depend on the diagonal section of its copula, those parameters are invariant under the diagonal splice operation. Therefore, in modelling problems this operation permits us to create different dependence structures with given tail dependence parameters.

Observe that, under the hypotheses of Theorem 7, we have that $C_1 \boxslash C_2$ and $C_2 \boxslash C_1$ are both copulas in $\mathbf{C}_\delta$ if, and only if, $C_1(u,v) + C_2^t(u,v) \le \delta(u) + \delta(v)$ on $[0,1]^2$ (equivalently, if $C_1^t(u,v) + C_2(u,v) \le \delta(u) + \delta(v)$ on $[0,1]^2$). As a consequence, the following corollary provides a sufficient condition to assure that both $C_1 \boxslash C_2$ and $C_2 \boxslash C_1$ are copulas:

Corollary 8. *Let δ be a diagonal, K_δ the copula given by (5), and C_1 and C_2 two copulas in $\mathbf{C}_\delta$ such that $\max(C_1(u,v), C_2(u,v)) \le K_\delta(u,v)$ for all (u,v) in $[0,1]^2$. Then, the diagonal splices $C_1 \boxslash C_2$ and $C_2 \boxslash C_1$ are both copulas in $\mathbf{C}_\delta$.*

Proof. Let (u,v) be in $[0,1]^2$. From the definition of K_δ, the inequality $\max(C_1(u,v), C_2(u,v)) \le K_\delta(u,v)$ is equivalent to the following: $\max(C_1(u,v), C_2(u,v)) \le (\delta(u)+\delta(v))/2$. So we have that $C_1(u,v) \le (\delta(u) + \delta(v))/2$ and $C_2(v,u) \le (\delta(v) + \delta(u))/2$ for every $(u,v) \in [0,1]^2$, whence $C_1(u,v) + C_2^t(u,v) \le \delta(u) + \delta(v)$ for every $(u,v) \in [0,1]^2$, and the conclusion follows. ■

As an immediate consequence of Corollary 8 and part (iv) of Theorem 2, we have the following result (which can be found also in Durante et al. (2005) with a different proof):

Corollary 9. *Let δ be a diagonal, and C_1 and C_2 two symmetric copulas in $\mathbf{C}_\delta$. Then, the diagonal splices $C_1 \boxslash C_2$ and $C_2 \boxslash C_1$ are copulas in $\mathbf{C}_\delta$.*

An application of the previous result is the following:

Example 10. Let δ be a diagonal, and let B_δ and K_δ be the copulas defined by (4) and (5). Then, the diagonal splices $B_\delta \boxslash K_\delta$ and $K_\delta \boxslash B_\delta$ are both copulas.

For instance, if δ is the diagonal of Example 5, i.e., $\delta(t) = \max(0, t - 1/3, 2t - 1)$ for all $t \in [0,1]$, we can obtain – after some computations – the copula $B_\delta \boxslash K_\delta$ as shown in Fig. 4, where the values of $B_\delta \boxslash K_\delta$ in various portions of $[0,1]^2$ are noted.

The next example shows that the conditions in Corollaries 8 and 9 are sufficient, but not necessary.

Example 11. Let C_1 be the copula whose probability mass is uniformly spread on the segments joining the points $(0,1)$ and $(1/3, 2/3)$, and the points $(1/3, 0)$ and $(1, 2/3)$, i.e., C_1 is the shuffle of Min given by: $C_1(u,v) = \min(u - 1/3, v)$ if $(u,v) \in [1/3, 1] \times [0, 2/3]$, and $C_1(u,v) = W(u,v)$ otherwise. Let C_2 be the copula of the same name in Example 5. Then it is easy to check that $\delta_{C_1}(t) = \delta_{C_2}(t)$, $C_1 \boxslash C_2 = C_1$ and $C_2 \boxslash C_1 = C_2$. Observe also that C_1 and C_2 are not symmetric, and that $C_1(2/3, 1/3) = C_2(2/3, 1/3) = 1/3 > 1/6 = K_\delta(2/3, 1/3)$.

The previous two examples provide singular asymmetric copulas. The copula diagonal splice method also provides a way to construct absolutely continuous asymmetric copulas with a given diagonal section. To show that, we need the following lemma:

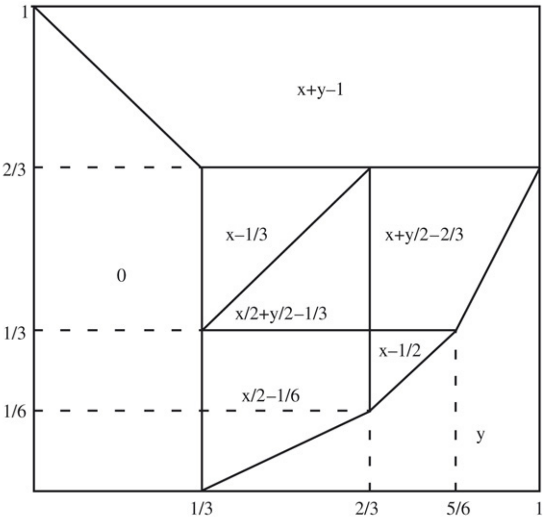

Fig. 4. The copula $B_\delta \boxslash K_\delta$ for $\delta(t) = \max(0, t - 1/3, 2t - 1)$.

Lemma 12. *Let c be a density whose support is in the square* $[0, 1]^2$*. Then, c is the density of a copula if and only if*

$$\int_0^1 c(u, v)\,\mathrm{d}v = 1 \quad \text{for all } u \in [0, 1], \tag{7}$$

and

$$\int_0^1 c(u, v)\,\mathrm{d}u = 1 \quad \text{for all } v \in [0, 1]. \tag{8}$$

Proof. If c is the density of a copula C, then $u = C(u, 1) = \int_0^u \int_0^1 c(x, v)\,\mathrm{d}v\,\mathrm{d}x$ for all $u \in [0, 1]$ and $v = C(1, v) = \int_0^v \int_0^1 c(u, y)\,\mathrm{d}u\,\mathrm{d}y$ for all $v \in [0, 1]$. Differentiating the last two equalities with respect to the variables u and v, respectively, we obtain (7) and (8). Conversely, let C be the distribution function associated with c. Then, $C(u, v) = \int_0^u \int_0^v c(x, y)\,\mathrm{d}y\,\mathrm{d}x$ for all $(u, v) \in [0, 1]^2$. To prove that C is a copula, it is sufficient to check that its margins are uniform on the interval $[0, 1]$. From (7) we have that $C(u, 1) = \int_0^u \int_0^1 c(x, v)\,\mathrm{d}v\,\mathrm{d}x = u$ for all $u \in [0, 1]$. Analogously, from (8) and Fubini's theorem we can obtain that $C(1, v) = v$ for all $v \in [0, 1]$, which completes the proof. ■

The following theorem characterizes those diagonal splices of absolutely continuous copulas with a common diagonal section which are themselves absolutely continuous copulas.

Theorem 13. *Let δ be a diagonal, and C_1 and C_2 two absolutely continuous copulas in $\mathbf{C}_\delta$ with densities c_1 and c_2, respectively. Then, the diagonal splice $C_1 \boxslash C_2$ is an absolutely continuous copula if, and only if, the following condition holds:*

$$\int_0^u c_1(u, v)\,\mathrm{d}v = \int_0^u c_2(u, v)\,\mathrm{d}v \quad \text{for all } u \in [0, 1]. \tag{9}$$

In this case, $C_1 \boxslash C_2$ is also in $\mathbf{C}_\delta$ and its density is equal to $c_1 \boxslash c_2$ almost everywhere.

Proof. Suppose first that condition (9) holds. Then, by integrating with respect to the variable u, we have

$$\int_0^1 \int_0^u c_1(u, v)\,\mathrm{d}v\,\mathrm{d}u = \int_0^1 \int_0^u c_2(u, v)\,\mathrm{d}v\,\mathrm{d}u,$$

i.e., both C_1 and C_2 spread the same probability mass on the triangle T_L (and hence, they also spread the same probability mass on T_U). From this fact, it is easy to obtain that

$$\int_0^1 \int_0^1 (c_1 \boxslash c_2)(u, v)\,\mathrm{d}v\,\mathrm{d}u = 1,$$

whence $c_1 \boxslash c_2$ is a density on $[0, 1]^2$. On the other hand, since δ is the diagonal section of C_1 and C_2 we can write, for $i = 1, 2$ and for every $t \in [0, 1]$, that

$$\delta(t) = \int_0^t \int_0^u c_i(u, v)\,\mathrm{d}v\,\mathrm{d}u + \int_0^t \int_0^v c_i(u, v)\,\mathrm{d}u\,\mathrm{d}v.$$

Differentiating the previous expression we have

$$\delta'(t) = \int_0^t c_i(t, v)\,\mathrm{d}v + \int_0^t c_i(u, t)\,\mathrm{d}u,$$

almost everywhere in $[0, 1]$, with $i = 1, 2$. Now, from this double equality and (9) we can obtain that

$$\int_0^v c_1(u, v)\,\mathrm{d}u = \int_0^v c_2(u, v)\,\mathrm{d}u \quad \text{for all } v \in [0, 1]. \tag{10}$$

We now prove that $c_1 \boxslash c_2$ is the density of a copula. From (9), we have that

$$\begin{aligned}\int_0^1 (c_1 \boxslash c_2)(u, v)\,\mathrm{d}v &= \int_0^u c_2(u, v)\,\mathrm{d}v + \int_u^1 c_1(u, v)\,\mathrm{d}v \\ &= \int_0^1 c_1(u, v)\,\mathrm{d}v = 1\end{aligned}$$

for every $u \in [0, 1]$; and from (10) we can also obtain that

$$\int_0^1 (c_1 \boxslash c_2)(u, v)\,\mathrm{d}u = 1$$

for every $v \in [0, 1]$. Thus, Lemma 12 assures that $c_1 \boxslash c_2$ is the density of a copula. Finally, by applying again (9) and (10), we can obtain that

$$\int_0^u \int_0^v (c_1 \boxslash c_2)(x, y)\,\mathrm{d}y\,\mathrm{d}x = (C_1 \boxslash C_2)(u, v),$$

for all $(u, v) \in [0, 1]^2$, i.e., $C_1 \boxslash C_2$ is an absolutely continuous copula with density $c_1 \boxslash c_2$.

Conversely, if $C_1 \boxslash C_2$ is an absolutely continuous copula, then its density should be almost everywhere equal to $c_1 \boxslash c_2$. Thus, from Lemma 12 we have

$$\int_0^1 (c_1 \boxslash c_2)(u, v)\,\mathrm{d}v = 1 = \int_0^1 c_1(u, v)\,\mathrm{d}v$$

for every $u \in [0, 1]$, whence condition (9) follows immediately. ■

If c is the density of a copula C, then it is easy to obtain that $\int_0^u c(u, v)\,\mathrm{d}v = (\partial C/\partial u)(u, u)$ for every $u \in [0, 1]$. Hence we have the following corollary, which is very useful since it provides a simpler characterization than that of Theorem 13.

Corollary 14. *Let C_1 and C_2 be two absolutely continuous copulas with a common diagonal section. Then, the diagonal splice $C_1 \boxslash C_2$ is an absolutely continuous copula if, and only if, the following condition holds:*

$$\frac{\partial C_1}{\partial u}(u,u) = \frac{\partial C_2}{\partial u}(u,u) \quad \text{for all } u \in [0,1]. \tag{11}$$

As an application of the previous results, in the following example we obtain a family of absolutely continuous asymmetric copulas from two families of *extreme value copulas*. Extreme value copulas have the following form:

$$C(u,v) = \exp\left\{\ln(uv) A\left(\frac{\ln v}{\ln(uv)}\right)\right\}, \qquad (u,v) \in (0,1]^2 \tag{12}$$

where A – called the *dependence function* of the extreme value copula C – is a convex function satisfying $A(0) = A(1) = 1$ and $\max(t, 1-t) \leq A(t) \leq 1$ (Joe, 1997; Nelsen, 2006). Observe that the diagonal section of a copula given by (12) is the function $\delta(t) = t^{2A(1/2)}$, $t \in [0,1]$. Now we provide the following example:

Example 15. Consider two families of extreme value copulas, those whose dependence functions are:

(1) $A_\theta(t) = \left(t^\theta + (1-t)^\theta\right)^{1/\theta}$, for every $\theta \geq 1$, which yield the so-called Gumbel–Hougaard family of copulas $\{C_\theta : \theta \geq 1\}$, and whose diagonal sections are given by $\delta_{C_\theta}(t) = t^{(2^{1/\theta})}$ for all $t \in [0,1]$.

(2) $A_\beta(t) = 1 - \beta t(1-t)$, for every $\beta \in [0,1]$, which yield a family of copulas $\{C_\beta : 0 \leq \beta \leq 1\}$ whose diagonal sections are given by $\delta_{C_\beta}(t) = t^{2-\beta/2}$ for all $t \in [0,1]$.

It is not difficult to check that all the copulas in these two families are absolutely continuous and symmetric, and that $\delta_{C_\theta} = \delta_{C_\beta}$ if $\beta \in [0,1]$ and $\theta = (\ln 2)/\ln(2-\beta/2)$. In this case, Corollary 9 assures that the diagonal splices $C_\theta \boxslash C_\beta$ and $C_\beta \boxslash C_\theta$ are both asymmetric copulas; moreover, from Corollary 14 we have that the copulas $C_\theta \boxslash C_\beta$ and $C_\beta \boxslash C_\theta$ are absolutely continuous.

Observe that the tail dependence parameters for $C_\theta \boxslash C_\beta$ and $C_\beta \boxslash C_\theta$ are the same as for C_θ and C_β, namely $\lambda_L = 0$ and $\lambda_U = \beta/2$ (which is positive since $\beta > 0$).

At this point, the examples of copulas obtained in this paper as a diagonal splice of two other copulas may appear as very special cases subject to restrictive conditions. However, the following example provides a family of absolutely continuous symmetric copulas such that the diagonal splice of any two copulas in that family is an absolutely continuous asymmetric copula.

Example 16. Let $\alpha \in [-3,1]$ and let C_α be the function defined by $C_\alpha(u,v) = uv\left[1 - \alpha(u-v)^2\right] + \alpha|u-v|(uv - \min(u^2, v^2))$ for all $(u,v) \in [0,1]^2$. After some computation, it can be proved that C_α is an absolutely continuous symmetric copula for every $\alpha \in [-3,1]$. Observe that $C_0 = \Pi$ and $\delta_{C_\alpha} = \delta_\Pi$ for every $\alpha \in [-3,1]$. Thus, from Corollary 9 we have that the diagonal splices $C_{\alpha_1} \boxslash C_{\alpha_2}$ and $C_{\alpha_2} \boxslash C_{\alpha_1}$ are copulas whenever $\alpha_1, \alpha_2 \in [-3,1]$. Moreover, since $(\partial C_\alpha/\partial u)(u,u) = u$ for all $u \in [0,1]$ and for every $\alpha \in [-3,1]$, we can conclude from Corollary 14 that both $C_{\alpha_1} \boxslash C_{\alpha_2}$ and $C_{\alpha_2} \boxslash C_{\alpha_1}$ are absolutely continuous asymmetric copulas in $\mathbf{C}_{\delta_\Pi}$ whenever α_1 and α_2 are two different numbers in $[-3,1]$.

The following example shows that the diagonal splice operation may not preserve the absolute continuity if condition (11) is not satisfied.

Example 17. Let C_1 be the copula with cubic horizontal and vertical sections (Nelsen, 2006; Nelsen et al., 1997) given by $C_1(u,v) = uv + uv(1-u)(1-v)(v-u)$ for all $(u,v) \in [0,1]^2$. Let $C_2 = C_1^t$. The copulas C_1 and C_2 are absolutely continuous. However, it is easy to check that the diagonal splices $C_1 \boxslash C_2$ and $C_2 \boxslash C_1$ are not absolutely continuous since both functions – $C_1 \boxslash C_2$ is a proper quasi-copula and $C_2 \boxslash C_1$ is a copula – have a singular component on the diagonal D: $C_1 \boxslash C_2$ concentrates a mass of $-1/15$ on D and $C_2 \boxslash C_1$ concentrates a mass of $1/15$ on D.

The copulas C_1 and C_2 in Example 17 do not spread the same probability mass on each of the triangles T_L and T_U. This is a necessary condition for the absolute continuity of $C_1 \boxslash C_2$ and $C_2 \boxslash C_1$, as shown in the proof of Theorem 13, but it is not sufficient, as the following example shows.

Example 18. Let C_1 and C_2 be the functions defined by $C_1(u,v) = uv + uv(1-u)(1-2u)(1-v)^2(1+v)$ and $C_2(u,v) = uv + uv(1-u)(1-v)[1 - v + v^2 - u(1 + 2v - 2v^2)]$ for every $(u,v) \in [0,1]^2$. By using Theorem 3.2.8 in Nelsen (2006) – see also Nelsen et al. (1997) – we have that C_1 and C_2 are absolutely continuous copulas with cubic horizontal sections. Moreover, $\delta_{C_1} = \delta_{C_2}$ and it is straightforward to verify that both C_1 and C_2 spread the same probability mass on each one of the triangles T_L and T_U (namely, a mass of $199/420$ on T_L and a mass of $221/420$ on T_U). However, the diagonal splices $C_1 \boxslash C_2$ and $C_2 \boxslash C_1$ are not absolutely continuous copulas since, as it is easy to check, condition (11) in Corollary 14 does not hold.

3. Further constructions and bounds for copulas with a given diagonal section

In this section we proceed with the study of two problems: the problem studied in Section 2 (construction of copulas and quasi-copulas with a given diagonal section through the diagonal splice method), and the problem of finding best-possible upper bounds for copulas with a given diagonal section.

Any set of copulas sharing a particular statistical property (for instance, copulas with the same diagonal section) is guaranteed to have pointwise best-possible bounds within the set of quasi-copulas $\mathbf{Q}$ (Nelsen et al., 2004). This is due to the fact that $(\mathbf{Q}, \preceq)$ is a complete lattice, which is order-isomorphic to the Dedekind–MacNeille completion of the set $\mathbf{C}$ of copulas (Nelsen and Úbeda-Flores, 2005).

The following result about best-possible bounds for copulas and quasi-copulas with a given diagonal section can be found in Nelsen et al. (2004).

Theorem 19. *Let δ be a diagonal. Then, the following statements hold:*

(i) *The best-possible lower bound for both the sets $\mathbf{C}_\delta$ and $\mathbf{Q}_\delta$ is the Bertino copula B_δ defined by (4).*
(ii) *The best-possible upper bound for the set $\mathbf{Q}_\delta$ is the quasi-copula A_δ defined by (6).*

But no general expression is known for the best-possible upper bound of the set $\mathbf{C}_\delta$, which we denote by $\overline{C}_\delta$. Since $(\mathbf{Q}, \leq)$ is a complete lattice, $\overline{C}_\delta$ is a quasi-copula, i.e., $\overline{C}_\delta \in \mathbf{Q}_\delta$. The following result provides other properties possessed by this quasi-copula.

Theorem 20. *Let δ be a diagonal, and let $\overline{C}_\delta$ be the best-possible upper bound of the set $\mathbf{C}_\delta$. Then $\overline{C}_\delta$ satisfies the following properties:*

(i) *$\overline{C}_\delta$ is symmetric.*
(ii) *$K_\delta \leq \overline{C}_\delta \leq A_\delta$.*
(iii) *$\overline{C}_\delta$ is a copula if and only if $\overline{C}_\delta = K_\delta$.*

Proof. Observe that the set $\mathbf{C}_\delta$ can also be seen as $\mathbf{C}_\delta = \{C^t : C \in \mathbf{C}_\delta\}$. Thus, for every $(u, v) \in [0, 1]^2$, $\overline{C}_\delta(v, u) = \sup(C(v, u) : C \in \mathbf{C}_\delta) = \sup(C^t(u, v) : C \in \mathbf{C}_\delta) = \overline{C}_\delta(u, v)$, i.e., property (i) holds. Since $K_\delta \in \mathbf{C}_\delta$, it is immediate that $K_\delta \leq \overline{C}_\delta$. Since $\overline{C}_\delta \in \mathbf{Q}_\delta$ and from part (ii) in Theorem 19, it is immediate that $\overline{C}_\delta \leq A_\delta$, whence property (ii) follows. Finally, property (iii) follows from part (iv) of Theorem 2. ■

With regard to part (iii) in the previous theorem, it is an open problem to characterize the diagonals δ such that $\overline{C}_\delta$ is a copula, i.e., $\overline{C}_\delta = K_\delta$.

At this point, we know $\overline{C}_\delta$ only for very few diagonals, those described in part (v) of Theorem 2. For those diagonals, the two inequalities in property (ii) of Theorem 20 are indeed equalities. However, in this section we prove that the equality $\overline{C}_\delta = A_\delta$ occurs for a rather large class of diagonals, including the diagonal sections of many copulas in the literature. The above-mentioned class of diagonals is introduced in the following definition:

Definition 21. A diagonal δ is said to be *simple* if $\hat{\delta}$ is a quasi-concave function, i.e., $\hat{\delta}(\alpha u + (1 - \alpha)v) \geq \min(\hat{\delta}(u), \hat{\delta}(v))$ for every $u, v, \alpha \in [0, 1]$.

The following result is a consequence of the above definition.

Theorem 22. *Let δ be a diagonal. Then, the following statements are equivalent:*

(i) *δ is simple.*
(ii) *The equality $l_\delta(u, v) = \min(\hat{\delta}(u), \hat{\delta}(v))$ holds for all $(u, v) \in [0, 1]^2$, i.e., the minimum of $\hat{\delta}$ on any closed interval $\mathcal{I} \subset [0, 1]$ is attained in one of the endpoints of $\mathcal{I}$.*
(iii) *There exists a point $c \in [0, 1]$ such that $\hat{\delta}$ is nondecreasing in $[0, c]$ and nonincreasing in $[c, 1]$.*

A probabilistic interpretation of the concept introduced in Definition 21 is the following: Let U and V be uniform [0, 1] random variables with associated copula C. Then, from Eq. (2), we have that the diagonal section of C is simple if and only if there exists $c \in [0, 1]$ such that the function $R(t) = \Pr[\min(U, V) \leq t < \max(U, V)]$ is nondecreasing in the interval $[0, c]$ and nonincreasing in $[c, 1]$.

The following result provides a subclass of the class of simple diagonals which appears frequently in the most commonly used families of copulas.

Theorem 23. *If δ is a convex diagonal, then δ is simple.*

Proof. From the definition of $\hat{\delta}$, it is clear that δ is convex if and only if $\hat{\delta}$ is concave. Let u, v be any points in [0, 1], and let $\alpha \in [0, 1]$. Since $\hat{\delta}$ is concave, we have that $\hat{\delta}(\alpha u + (1 - \alpha)v) \geq \alpha\hat{\delta}(u) + (1 - \alpha)\hat{\delta}(v) \geq \alpha \min(\hat{\delta}(u), \hat{\delta}(v)) + (1 - \alpha) \min(\hat{\delta}(u), \hat{\delta}(v)) = \min(\hat{\delta}(u), \hat{\delta}(v))$. Therefore, the diagonal δ is simple. ■

The following examples illustrate Definition 21 and Theorems 22 and 23. Part (2) in these examples also shows that the converse of Theorem 23 does not hold.

Example 24. Here we examine some examples of diagonals:

(1) The diagonal sections of the copulas Π, W and M are convex, and therefore simple (recall that the graphs of the functions δ_Π and $\hat{\delta}_\Pi$ have been drawn in Fig. 1).
(2) The function $\delta_1(t) = \min(t, \max(1/2, 2t - 1))$, with $t \in [0, 1]$, is a diagonal such that $\hat{\delta}_1(t) = \min(\max(0, t - 1/2), 1 - t)$ for all $t \in [0, 1]$. So δ_1 is simple but not convex (see the first graph in Fig. 5).
(3) Let δ_2 be the diagonal given by $\delta_2(t) = \min(\max(0, 2t - 1/2), \max(1/2, 2t - 1))$, for all $t \in [0, 1]$. Then, $\hat{\delta}_2(t) = \max(\min(t, 1/2 - t), \min(t - 1/2, 1 - t))$ for all $t \in [0, 1]$. So δ_2 is not simple (see the second graph in Fig. 5).

In addition to the copulas in part (1) of Example 24, many other copulas have simple diagonal sections. For instance, it can be checked that all the members of the 22 families of Archimedean copulas presented in Nelsen (2006) have simple diagonal sections (moreover, the diagonal sections are convex in 21 of those families, with the exception of the family 4.2.18). However, not every Archimedean copula has a simple diagonal section. For instance, the following example provides a family of Archimedean copulas so that we can find among them an infinite number of copulas whose diagonal sections are not simple.

Example 25. Let $\alpha \in [1, \infty)$, and let φ_α be the function defined on [0, 1] by $\varphi_\alpha(t) = \max(\alpha - (2\alpha - 1)t, 1 - t)$. Since φ_α is a continuous, strictly decreasing and convex function such that $\varphi_\alpha(1) = 0$, we have that φ_α is a generator of an Archimedean copula C_α for every $\alpha \in [1, \infty)$ (Nelsen, 2006). After simple computations, for each $\alpha \in [1, \infty)$ we obtain that the diagonal section of the Archimedean copula C_α is the function given by $\delta_{C_\alpha}(t) = 0$, if $t \in [0, \alpha/(4\alpha - 2)]$; $\delta_{C_\alpha}(t) = 2t - \alpha/(2\alpha - 1)$, if $t \in [\alpha/(4\alpha - 2), 1/2]$; $\delta_{C_\alpha}(t) = (2t + \alpha - 2)/(2\alpha - 1)$, if $t \in [1/2, 3/4]$; and $\delta_{C_\alpha}(t) = 2t - 1$, if

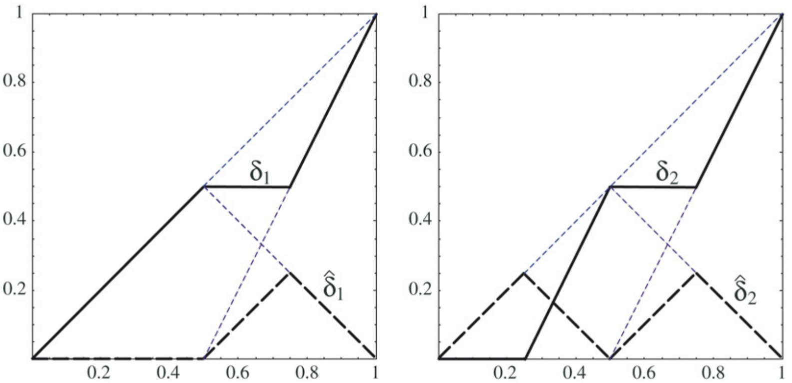

Fig. 5. The functions δ_1, $\hat{\delta}_1$, δ_2 and $\hat{\delta}_2$ in Example 24.

$t \in [3/4, 1]$. Observe finally that, if $\alpha > 3/2$, then $\hat{\delta}_{C_\alpha}(1/2) = 1/(4\alpha - 2) < \min(\hat{\delta}_{C_\alpha}(\alpha/(4\alpha - 2)), \hat{\delta}_{C_\alpha}(3/4)) = 1/4$, whence δ_{C_α} is not simple whenever $\alpha > 3/2$.

In order to study the two problems mentioned at the beginning of this section, we need four preliminary lemmas. In the first one, for any diagonal δ we prove a property of the function h_δ defined by (3). Other properties of that function can be found in Úbeda-Flores (2001).

Lemma 26. *Let δ be a diagonal, and let h_δ be the function defined by* (3). *Then,* $|h_\delta(u_2, v_2) - h_\delta(u_1, v_1)| \le |u_2 - u_1| + |v_2 - v_1|$ *for every* $(u_1, v_1), (u_2, v_2)$ *in* $[0, 1]^2$.

Proof. Since h_δ is symmetric, we only need to prove that

$$|h_\delta(u_2, v) - h_\delta(u_1, v)| \le u_2 - u_1 \tag{13}$$

whenever $u_1, u_2, v \in [0, 1]$ and $u_1 < u_2$. We consider three cases:

(1) $u_1 < u_2 \le v$. In this case, $|h_\delta(u_2, v) - h_\delta(u_1, v)| = h_\delta(u_1, v) - h_\delta(u_2, v)$. Let $t_0 \in [u_1, v]$ be such that $h_\delta(u_1, v) = \hat{\delta}(t_0)$. If $t_0 \in [u_2, v]$, then the inequality (13) is trivially satisfied; otherwise, if $t_0 \in [u_1, u_2)$, then, from property (ii) of Theorem 1 and the definition of h_δ, we obtain that $h_\delta(u_1, v) - h_\delta(u_2, v) \le \hat{\delta}(t_0) - \hat{\delta}(u_2) \le u_2 - t_0 \le u_2 - u_1$.

(2) $u_1 \le v < u_2$. Let $t_1 \in [u_1, v]$ and $t_2 \in [v, u_2]$ be such that $h_\delta(u_1, v) = \hat{\delta}(t_1)$ and $h_\delta(u_2, v) = \hat{\delta}(t_2)$. Now, a similar reasoning to that in case (1) leads to $|h_\delta(u_2, v) - h_\delta(u_1, v)| = |\hat{\delta}(t_2) - \hat{\delta}(t_1)| \le t_2 - t_1 \le u_2 - u_1$.

(3) $v < u_1 < u_2$. Let $t_0 \in [v, u_2]$ be such that $h_\delta(u_2, v) = \hat{\delta}(t_0)$. Analogously to case (1), the inequality (13) is trivial when $t_0 \in [v, u_1]$; and, if $t_0 \in (u_1, u_2]$, then $|h_\delta(u_2, v) - h_\delta(u_1, v)| \le \hat{\delta}(t_0) - \hat{\delta}(u_1) \le t_0 - u_1 \le u_2 - u_1$, which completes the proof. ■

The following lemma exhibits a property of h_δ when δ is a simple diagonal.

Lemma 27. *Let δ be a simple diagonal. Then the function h_δ defined by* (3) *is 2-decreasing in both the sets T_U and T_L.*

Proof. Since h_δ is symmetric, it suffices to prove that h_δ is 2-decreasing in T_U. Let $J = [u_1, u_2] \times [v_1, v_2]$ be any 2-box in T_U, i.e., $u_1 \le u_2 \le v_1 \le v_2$. We have to prove that $V_{h_\delta}(J) = h_\delta(u_2, v_2) - h_\delta(u_2, v_1) - h_\delta(u_1, v_2) + h_\delta(u_1, v_1) \le 0$. Let $\alpha \in [u_2, v_2]$, $\beta \in [u_2, v_1]$, $\gamma \in [u_1, v_2]$ and $\epsilon \in [u_1, v_1]$ such that $h_\delta(u_2, v_2) = \hat{\delta}(\alpha)$, $h_\delta(u_2, v_1) = \hat{\delta}(\beta)$, $h_\delta(u_1, v_2) = \hat{\delta}(\gamma)$ and $h_\delta(u_1, v_1) = \hat{\delta}(\epsilon)$. Then $V_{h_\delta}(J) = \hat{\delta}(\alpha) - \hat{\delta}(\beta) - \hat{\delta}(\gamma) + \hat{\delta}(\epsilon)$; and, from the definition of h_δ, we have $\hat{\delta}(\beta) \le \min(\hat{\delta}(\alpha), \hat{\delta}(\epsilon)) \le \max(\hat{\delta}(\alpha), \hat{\delta}(\epsilon)) \le \hat{\delta}(\gamma)$. There are three cases to consider:

(1) If $\alpha \in [u_2, v_1]$, then $\hat{\delta}(\alpha) = \hat{\delta}(\beta)$ and $V_{h_\delta}(J) = \hat{\delta}(\epsilon) - \hat{\delta}(\gamma) \le 0$.

(2) If $\epsilon \in [u_2, v_1]$, then $\hat{\delta}(\epsilon) = \hat{\delta}(\beta)$ and $V_{h_\delta}(J) = \hat{\delta}(\alpha) - \hat{\delta}(\gamma) \le 0$.

(3) If $\epsilon \in [u_1, u_2)$ and $\alpha \in (v_1, v_2]$, then $\epsilon < \beta < \alpha$ and $\hat{\delta}(\beta) < \min(\hat{\delta}(\alpha), \hat{\delta}(\epsilon))$, contrary to the hypothesis that δ is simple; so only the cases (1) and (2) are possible, which completes the proof. ■

The following result proves the 2-increasingness of A_δ in both the sets T_U and T_L when δ is a simple diagonal.

Lemma 28. *Let δ be a simple diagonal, and let A_δ be the quasi-copula defined by* (6). *Then A_δ is 2-increasing in both T_U and T_L.*

Proof. Since A_δ is symmetric, it suffices to prove that A_δ is 2-increasing in T_U. Let $J = [u_1, u_2] \times [v_1, v_2]$ be a 2-box included in T_U. Then, $V_{A_\delta}(J) = \min(u_2, v_2 - h_\delta(u_2, v_2)) - \min(u_2, v_1 - h_\delta(u_2, v_1)) - \min(u_1, v_2 - h_\delta(u_1, v_2)) + \min(u_1, v_1 - h_\delta(u_1, v_1))$. We study the following two cases:

(1) $u_1 \le v_1 - h_\delta(u_1, v_1)$. From Lemma 26, we have $|h_\delta(u_1, v_2) - h_\delta(u_1, v_1)| = h_\delta(u_1, v_2) - h_\delta(u_1, v_1) \le v_2 - v_1$, whence $v_1 - h_\delta(u_1, v_1) \le v_2 - h_\delta(u_1, v_2)$. So $A_\delta(u_1, v_1) = A_\delta(u_1, v_2) = u_1$, and therefore $V_{A_\delta}(J) = A_\delta(u_2, v_2) - A_\delta(u_2, v_1) \ge 0$.

(2) $v_1 - h_\delta(u_1, v_1) < u_1$. In this case, $V_{A_\delta}(J) \ge A_\delta(u_2, v_2) - (v_1 - h_\delta(u_2, v_1)) - A_\delta(u_1, v_2) + v_1 - h_\delta(u_1, v_1) = A_\delta(u_2, v_2) - A_\delta(u_1, v_2) + h_\delta(u_2, v_1) - h_\delta(u_1, v_1)$. We need to consider two subcases here:

   (2a) $u_2 \le v_2 - h_\delta(u_2, v_2)$. In this case, from Lemma 26 we obtain that $V_{A_\delta}(J) \ge u_2 - u_1 + h_\delta(u_2, v_1) - h_\delta(u_1, v_1) \ge 0$.

(2b) $v_2 - h_\delta(u_2, v_2) < u_2$. In this case, from Lemma 27 we have $V_{A_\delta}(J) \ge v_2 - h_\delta(u_2, v_2) - (v_2 - h_\delta(u_1, v_2)) + h_\delta(u_2, v_1) - h_\delta(u_1, v_1) = -V_{h_\delta}(J) \ge 0$, which completes the proof. ■

In general, the converse of Lemma 28 is not true, as the following example shows.

Example 29. Let δ_2 be the nonsimple diagonal considered in part (3) of Example 24. However, A_{δ_2} is a copula from part (v) in Theorem 2; so that A_{δ_2} is 2-increasing in both T_U and T_L.

In the following lemma, we present a method for constructing copulas from a certain class of quasi-copulas with simple diagonal sections.

Lemma 30. *Let δ be a simple diagonal, and B_δ the copula defined by (4). If Q is a quasi-copula in $\mathbf{Q}_\delta$, and is 2-increasing in T_U (respectively T_L), then the diagonal splice $Q \boxslash B_\delta$ (respectively $B_\delta \boxslash Q$) is a copula in $\mathbf{C}_\delta$.*

Proof. We prove the case where Q is 2-increasing in T_U: the proof of the other case is analogous. A similar reasoning to that in the proof of Theorem 7 leads to the following equivalence: the diagonal splice $Q \boxslash B_\delta$ is in $\mathbf{C}_\delta$ if, and only if, $V_{Q \boxslash B_\delta}([u, v]^2) \ge 0$ for every $(u, v) \in T_U \setminus D$. From the hypothesis and from part (ii) of Theorem 19, we have that $V_{Q \boxslash B_\delta}([u, v]^2) = \delta(v) - Q(u, v) - B_\delta(v, u) + \delta(u) \ge \delta(v) - A_\delta(u, v) - B_\delta(v, u) + \delta(u) = v - \hat{\delta}(v) - \min(u, v - h_\delta(u, v)) + l_\delta(v, u) - \hat{\delta}(u) \ge v - \hat{\delta}(v) - v + h_\delta(u, v) + \min(\hat{\delta}(u), \hat{\delta}(v)) - \hat{\delta}(u) = h_\delta(u, v) - \max(\hat{\delta}(u), \hat{\delta}(v)) \ge 0$, which completes the proof. ■

As an immediate consequence of Lemmas 28 and 30 we have the following result:

Theorem 31. *Let δ be a simple diagonal, and let B_δ and A_δ be the functions defined by (4) and (6), respectively. Then, the diagonal splices $A_\delta \boxslash B_\delta$ and $B_\delta \boxslash A_\delta$ are both copulas in $\mathbf{C}_\delta$.*

As an application of Theorem 31, the following result provides the best-possible upper bound for the set $\mathbf{C}_\delta$ when δ is a simple diagonal.

Theorem 32. *Let δ be a simple diagonal. Then, the best-possible upper bound for the set $\mathbf{C}_\delta$ is $\overline{C}_\delta = A_\delta$.*

Proof. From Theorem 31, we have that both the diagonal splices $A_\delta \boxslash B_\delta$ and $B_\delta \boxslash A_\delta$ are copulas in $\mathbf{C}_\delta$. Thus, $\overline{C}_\delta(u, v) \ge \max((A_\delta \boxslash B_\delta)(u, v), (B_\delta \boxslash A_\delta)(u, v)) = A_\delta(u, v)$. Since $\overline{C}_\delta \le A_\delta$, the conclusion follows. ■

The converses of Theorems 31 and 32 do not hold, in the sense that, either $A_\delta \boxslash B_\delta$ and $B_\delta \boxslash A_\delta$ could be copulas in $\mathbf{C}_\delta$, or $\overline{C}_\delta = A_\delta$, and δ not simple. For example, the diagonal δ_2 considered in part (3) of Example 24 and in Example 29 is not simple; but part (v) of Theorem 2, Corollary 9 and part (ii) of Theorem 20 yield that the diagonal splices $A_{\delta_2} \boxslash B_{\delta_2}$ and $B_{\delta_2} \boxslash A_{\delta_2}$ are copulas in $\mathbf{C}_{\delta_2}$, and that $\overline{C}_{\delta_2} = A_{\delta_2}$. On the other hand, we can also find examples where $A_\delta \boxslash B_\delta$ and/or $B_\delta \boxslash A_\delta$ are proper quasi-copulas, and examples where the equality $\overline{C}_\delta = A_\delta$ does not hold, as the following example shows.

Example 33. Let δ be the diagonal section given by $\delta(t) = \min(\max(0, 2t - 2/3), \max(1/3, 2t - 1))$ for all $t \in [0, 1]$.

(A) It can be computed that $A_\delta(u, v) = \min(1/3, u + v - 2/3)$ if $(u, v) \in [1/3, 2/3]^2$, $A_\delta(u, v) = \max(0, u - 1/3, v - 1/3, u + v - 1)$ if $-1/3 \le v - u \le 1/3$ and $(u, v) \notin [1/3, 2/3]^2$, and $A_\delta(u, v) = M(u, v)$ otherwise. From part (v) of Theorem 2 we have that A_δ is a proper quasi-copula, and it can be checked that A_δ spreads uniformly a mass of $-1/3$ on the segment joining the points $(1/3, 2/3)$ and $(2/3, 1/3)$, and a mass of $2/3$ on each one of the segments joining the points $(0, 1/3)$ and $(2/3, 1)$, and the points $(1/3, 0)$ and $(1, 2/3)$. Hence it is easy to conclude that $A_\delta \boxslash B_\delta$ and $B_\delta \boxslash A_\delta$ are proper quasi-copulas.

(B) If C is any copula in $\mathbf{C}_\delta$, then $V_C([1/2, 2/3] \times [0, 2/3]) = \delta(2/3) - C(1/2, 2/3)$ and $V_C([1/3, 1] \times [1/3, 1/2]) = 1/2 - 1/3 - C(1/3, 1/2) + \delta(1/3)$, whence $C(1/2, 2/3) \le \delta(2/3) = 1/3$ and $C(1/3, 1/2) \le 1/6$. Furthermore, $V_C([1/3, 1/2] \times [1/2, 2/3]) = C(1/2, 2/3) - \delta(1/2) - C(1/3, 2/3) + C(1/3, 1/2)$, and hence, by using the last two inequalities, we obtain that $C(1/3, 2/3) \le C(1/2, 2/3) + C(1/3, 1/2) - \delta(1/2) \le 1/3 + 1/6 - 1/3 = 1/6$. Thus, $\overline{C}_\delta(1/3, 2/3) \le 1/6$, but $A_\delta(1/3, 2/3) = 1/3$. Therefore $\overline{C}_\delta \ne A_\delta$.

References

Abdous, B., Fougères, A.-L., Ghoudi, K., 2005. Extreme behaviour for bivariate elliptical distributions. The Canadian Journal of Statistics 33, 317–334.

Ané, T., Kharoubi, C., 2003. Dependence structure and risk measure. Journal of Business 76, 411–438.

Alsina, C., Nelsen, R.B., Schweizer, B., 1993. On the characterization of a class of binary operations on distribution functions. Statistics & Probability Letters 17, 85–89.

Bäuerle, N., Müller, A., 1998. Modelling and comparing dependencies in multivariate risk portfolios. Astin Bulletin 28, 59–67.

Bertino, S., 1977. Sulla dissomiglianza tra mutabili cicliche. Metron 35, 53–88.

Denuit, M., Dhaene, J., Goovaerts, M., Kaas, R., 2005. Actuarial Theory for Dependent Risk: Measures, Orders and Models. John Wiley & Sons.

Durante, F., Mesiar, R., Sempi, C., 2005. Copulas with given diagonal section: Some new results. In: Proceedings of EUSFLAT–LFA Conference. Barcelona, pp. 931–936.

Durante, F., Kolesárová, A., Mesiar, R., Sempi, C., 2007. Copulas with given diagonal sections: Novel constructions and applications. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 15, 397–410.

Embrechts, P., McNeil, A.J., Straumann, D., 2002. Correlation and dependence in risk management: Properties and pitfalls. In: Risk Management: Value at Risk and Beyond. Cambridge University Press, Cambridge, pp. 176–223.

Frahm, G., Junker, M., Schmidt, R., 2005. Estimating the tail dependence coefficient: Properties and pitfalls. Insurance: Mathematics and Economics 37, 80–100.

Frahm, G., Junker, M., Szimayer, A., 2003. Elliptical copulas: Applicability and limitations. Statistics & Probability Letters 63, 275–286.

Fredricks, G.A., Nelsen, R.B., 1997. Copulas constructed from diagonal sections. In: Distributions with Given Marginals and Moment Problems. Kluwer, Dordrecht, pp. 129–136.

Fredricks, G.A., Nelsen, R.B., 2002. The Bertino family of copulas. In: Distributions with Given Marginals and Statistical Modeling. Kluwer, Dordrecht, pp. 81–92.

Frees, E.W., Valdez, E., 1998. Understanding relationships using copulas. North American Actuarial Journal 2, 1–25.

Genest, C., Quesada-Molina, J.J., Rodríguez-Lallena, J.A., Sempi, C., 1999. A characterization of quasi-copulas. Journal of Multivariate Analysis 69, 193–205.

Joe, H., 1997. Multivariate Models and Dependence Concepts. Chapman & Hall, London.

Klugman, S.A., Parsa, R., 1999. Fitting bivariate loss distributions with copulas. Insurance: Mathematics and Economics 24, 139–148.

Malevergne, Y., Sornette, D., 2006. Extreme Financial Risks. From Dependence to Risk Management. Springer-Verlag, Berlin.

Mikusiński, P., Sherwood, H., Taylor, M.D., 1992. Shuffles of Min. Stochastica 13, 61–74.

Nelsen, R.B., 2006. An Introduction to Copulas, Second ed. Springer, New York.

Nelsen, R.B., Fredricks, G.A., 1997. Diagonal copulas. In: Distributions with Given Marginals and Moment Problems. Kluwer, Dordrecht, pp. 121–128.

Nelsen, R.B., Quesada-Molina, J.J., Rodríguez-Lallena, J.A., 1997. Bivariate copulas with cubic sections. Journal of Nonparametric Statistics 7, 205–220.

Nelsen, R.B., Quesada-Molina, J.J., Rodríguez-Lallena, J.A., Úbeda-Flores, M., 2001. Distribution functions of copulas: A class of bivariate probability integral transforms. Statistics & Probability Letters 54, 277–282.

Nelsen, R.B., Quesada-Molina, J.J., Rodríguez-Lallena, J.A., Úbeda-Flores, M., 2004. Best-possible bounds on sets of bivariate distribution functions. Journal of Multivariate Analysis 90, 348–358.

Nelsen, R.B., Quesada-Molina, J.J., Schweizer, B., Sempi, C., 1996. Derivability of some operations on distribution functions. In: Distributions with Fixed Marginals and Related Topics. In: Lecture Notes–Monograph Series, vol. 28. Institute of Mathematical Statistics, Hayward, CA, pp. 233–243.

Nelsen, R.B., Úbeda-Flores, M., 2005. The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas. Comptes Rendus Mathématique 341, 583–586.

Rodríguez-Lallena, J.A., Úbeda-Flores, M., 2004. Best-possible bounds on sets of multivariate distribution functions. Communications in Statistics–Theory and Methods 33, 805–820.

Schmidt, R., 2002. Tail dependence for elliptically contoured distributions. Mathematical Methods of Operations Research 55, 301–327.

Sklar, A., 1959. Fonctions de répartition à *n* dimensions et leurs marges. Publications de l'Institut de Statistique de l'Université de Paris 8, 229–231.

Úbeda-Flores, M., 2001. Cópulas y cuasicópulas: interrelaciones y nuevas propiedades. Aplicaciones (Ph.D. Dissertation). Servicio de Publicaciones de la Universidad de Almería, Spain.