

Quasi-copulas and signed measures

Roger B. Nelsen^a, José Juan Quesada-Molina^b, José Antonio Rodríguez-Lallena^c,
Manuel Úbeda-Flores^{c,*}

^aDepartment of Mathematical Sciences, Lewis & Clark College, Portland, OR 97219, USA

^bDepartamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain

^cDepartamento de Estadística y Matemática Aplicada, Universidad de Almería, Carretera de Sacramento s/n,
04120 La Cañada de San Urbano, Almería, Spain

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Abstract

We study the relationship between multivariate quasi-copulas and measures that they may or may not induce on $[0, 1]^n$. We first study the mass distribution of the pointwise best possible lower bound for the set of n -quasi-copulas for $n \geq 3$. As a consequence, we show that not every n -quasi-copula induces a signed measure on $[0, 1]^n$.

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1. Introduction and preliminaries

Aggregation of pieces of information coming from different sources is an important task in expert and decision support systems, multi-criteria decision making, and group decision making. Aggregation operators are precisely the mathematical objects that allow this type of information fusion; and well-known operations in logic, probability theory, and statistics fit into this concept (for an overview, see [2,5,19]). Conjunctive aggregation operators [15], i.e., those aggregation operators that are bounded by the minimum, constitute an important class of operators that includes copulas and quasi-copulas.

Quasi-copulas were introduced in the field of probability (see [1,27]; and for the characterization of quasi-copula which now is usually utilized as definition, see [6,17]). They are also used in aggregation processes because they ensure that the aggregation is stable, in the sense that small error inputs correspond to small error outputs. In the few last years an increasing interest has been devoted to these functions by researchers in some topics of fuzzy sets theory, such as preference modeling, similarities and fuzzy logics: see, for instance, [7,9–14,16,20,22,30].

Let $n \geq 2$ be a natural number. An n -dimensional *quasi-copula* (briefly n -quasi-copula; or simply quasi-copula, if it is not necessary to specify the dimension) is a function Q from \mathbb{I}^n to \mathbb{I} ($\mathbb{I} = [0, 1]$) satisfying the following

* Corresponding author.

E-mail addresses: nelsen@lclark.edu (R.B. Nelsen), jquesada@ugr.es (J.J. Quesada-Molina), jarodrig@ual.es (J.A. Rodríguez-Lallena), mubeda@ual.es (M. Úbeda-Flores).

conditions:

- (Q1) *Boundary conditions*: For every $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{I}^n$, $Q(\mathbf{u})=0$ if at least one coordinate of \mathbf{u} is equal to 0; and $Q(\mathbf{u}) = u_k$ whenever all coordinates of \mathbf{u} are equal to 1 except maybe u_k .
- (Q2) *Monotonicity*: Q is nondecreasing in each variable.
- (Q3) *Lipschitz condition*: For every $\mathbf{u}, \mathbf{v} \in \mathbb{I}^n$, it holds that $|Q(\mathbf{u}) - Q(\mathbf{v})| \leq \sum_{i=1}^n |u_i - v_i|$.

The term *copula*, coined by Sklar [31,32], is now common in the statistical literature. An n -copula is a function C from \mathbb{I}^n to \mathbb{I} that satisfies condition (Q1) for n -quasi-copulas and, in place of (Q2) and (Q3), the stronger condition:

- (Q4) *The n -increasing property*: $V_C(B) = \sum (-1)^{k(\mathbf{c})} C(\mathbf{c}) \geq 0$ for every n -box $B = \times_{i=1}^n [a_i, b_i]$ in \mathbb{I}^n , where the sum is taken over all the *vertices* $\mathbf{c} = (c_1, c_2, \dots, c_n)$ of B (i.e., each c_k is equal to either a_k or b_k) and $k(\mathbf{c})$ is the number of indices k 's such that $c_k = a_k$.

Thus, every n -copula is an n -quasi-copula. A *proper n -quasi-copula* is an n -quasi-copula which is not an n -copula. The number $V_C(B)$ is usually called the *C -volume* of B , a concept that can be extended to n -quasi-copulas. Every n -quasi-copula Q satisfies the following condition:

$$W^n(\mathbf{u}) = \max \left(\sum_{i=1}^n u_i - n + 1, 0 \right) \leq Q(\mathbf{u}) \leq \min(u_1, u_2, \dots, u_n) = M^n(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathbb{I}^n \tag{1.1}$$

(a superscript on the name of a copula or quasi-copula denotes dimension rather than exponentiation). It is known that (a) M^n is an n -copula for every $n \geq 2$, (b) W^2 is a 2-copula, and (c) W^n is a proper n -quasi-copula for every $n \geq 3$. For a detailed study on copulas and an introduction to quasi-copulas, see [24].

In the literature several interesting similarities and differences between copulas and proper quasi-copulas have been shown (see, for instance, [8,23–26,29]). In this paper we show some new similarities and differences with respect to the type of the measure induced by copulas and proper quasi-copulas on \mathbb{I}^n .

It is known that every n -copula C induces a *stochastic measure* μ_C defined on the Lebesgue σ -algebra for \mathbb{I}^n , i.e., μ_C is a measure such that $\mu_C(\mathbb{I}^{i-1} \times A \times \mathbb{I}^{n-i}) = \lambda_1(A)$, for every $i = 1, 2, \dots, n$ and every Lebesgue measurable set A in \mathbb{I} , where λ_1 denotes the Lebesgue measure in \mathbb{R} . The stochastic measure μ_C is characterized by the fact that $\mu_C(B) = V_C(B)$ for every n -box B in \mathbb{I}^n . We wonder whether those results about copulas and measures might be generalized to proper quasi-copulas and signed measures. An almost complete answer to this question is given in the following two sections.

We finish these preliminaries by recalling the concept and some basic results on signed measures (for more details see, for instance, [21]). A *signed measure* μ on a measurable space (S, \mathcal{A}) is an extended real valued, countably additive set function on the σ -algebra \mathcal{A} such that $\mu(\emptyset) = 0$ and μ assumes at most one of the values ∞ and $-\infty$. Equivalently, a signed measure is the difference between two measures μ_1 and μ_2 such that at least one of them is finite. If μ is a signed measure on a measurable space (S, \mathcal{A}) , then there exist two disjoint sets D_+ and D_- whose union is S , and such that $\mu(E \cap D_+) \geq 0$ and $\mu(E \cap D_-) \leq 0$ for every $E \in \mathcal{A}$. The sets D_+ and D_- are said to form a *Hahn decomposition* of S with respect to μ . As for positive measures, we will say that a signed measure μ , defined on the Lebesgue σ -algebra for \mathbb{I}^n , is *stochastic* if $\mu(\mathbb{I}^{i-1} \times A \times \mathbb{I}^{n-i}) = \lambda_1(A)$, for every $i = 1, 2, \dots, n$ and every Lebesgue measurable set A in \mathbb{I} .

2. Proper quasi-copulas and measures

Let $n \geq 2$ be a natural number, and let Q be an n -quasi-copula. If \mathcal{B} denotes the algebra generated by the n -boxes in \mathbb{I}^n , it is clear that there exists a unique finitely additive (possibly signed) measure m_Q such that $m_Q(B) = V_C(B)$ for every n -box B in \mathbb{I}^n (and, as a consequence, $m_C(\mathbb{I}^{i-1} \times [a, b] \times \mathbb{I}^{n-i}) = \lambda_1([a, b]) = b - a$, for every $i = 1, 2, \dots, n$ and every closed interval $[a,b]$ in \mathbb{I}). For a detailed study on finitely additive (signed or not) measures, also called *charges*, see [4].

If Q is an n -copula it is known that the finitely additive measure m_Q could be extended to a *stochastic measure* μ_Q defined on the Lebesgue σ -algebra for \mathbb{I}^n . Our goal is to know whether the finitely additive signed measure m_Q associated with any proper n -quasi-copula Q could be extended to a *stochastic signed measure* μ_Q defined on the Lebesgue σ -algebra for \mathbb{I}^n .

We divide our study into two cases: the bivariate case and the multivariate case.

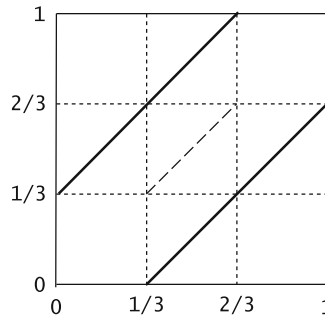


Fig. 1. The set where the 2-quasi-copula Q in Example 2.1 spreads positive mass (two segments) and negative mass (the dashed segment).

2.1. Proper bivariate quasi-copulas and stochastic signed measures

As far as we know, every proper bivariate quasi-copula appearing in the literature induces a stochastic signed measure on \mathbb{I}^2 . The following example illustrates this fact.

Example 2.1. Let s_1, s_2 and s_3 be three segments in \mathbb{I}^2 , given by the graphs of the following three functions, respectively (see Fig. 1): $f_1(x) = x + 1/3$ for every $x \in [0, 2/3]$; $f_2(x) = x$ for every $x \in [1/3, 2/3]$; and $f_3(x) = x - 1/3$ for every $x \in [1/3, 1]$. We spread mass $2/3$ uniformly on each of s_1 and s_3 , and mass $-1/3$ uniformly on s_2 . Let (u, v) be a point in \mathbb{I}^2 . If we define $Q(u, v)$ as the net mass in the 2-box $[0, u] \times [0, v]$, then Q is a proper 2-quasi-copula, specifically $Q(u, v) = \min(u, v, \max(0, u + v - 1, u - 1/3, v - 1/3))$ for every (u, v) in \mathbb{I}^2 . Furthermore, there exists a stochastic signed measure μ_Q defined on the Lebesgue σ -algebra for \mathbb{I}^2 such that $\mu_Q(B) = V_Q(B)$ for every 2-box B in \mathbb{I}^2 (which is the difference between the positive measure μ_Q^+ obtained by spreading mass $2/3$ uniformly on each of s_1 and s_3 , and the positive measure μ_Q^- obtained by spreading mass $1/3$ uniformly on s_2).

Nevertheless, we have not been able to prove or disprove whether every proper 2-quasi-copula induces a stochastic signed measure on \mathbb{I}^2 . In [26], we have found sequences of proper 2-quasi-copulas which induce stochastic signed measures on \mathbb{I}^2 with as much negative (and positive) mass as desired. But the limits of those sequences are also 2-quasi-copulas which induce stochastic (signed or not) measures on \mathbb{I}^2 . We have not been able to find 2-quasi-copulas which do not induce stochastic signed measures on \mathbb{I}^2 , i.e., 2-quasi-copulas which spread infinite negative (and positive) mass on \mathbb{I}^2 . However, we have a conclusive answer to that question for the multivariate case, as shown in the following subsection.

2.2. Proper multivariate quasi-copulas and stochastic signed measures

Let $n \geq 3$ be a natural number. Certainly, many proper n -quasi-copulas induce stochastic signed measures on the Lebesgue σ -algebra for \mathbb{I}^n in the above-mentioned way. To illustrate this fact, we provide the following example.

Example 2.2. Let Q be the function given by $Q(u, v, w) = \min(uv, uv, vw)$, $(u, v, w) \in \mathbb{I}^3$. Q is a proper 3-quasi-copula (see [28]). Now we will show that Q assigns mass $1/2$ uniformly to each of three triangles in \mathbb{I}^3 , namely, one with vertices $(0,0,0), (1,1,0), (1,1,1)$ (T_1), one with vertices $(0,0,0), (1,0,1), (1,1,1)$ (T_2) and one with vertices $(0,0,0), (0,1,1), (1,1,1)$ (T_3); and mass $-1/2$ (but not uniformly) on the line segment S connecting the point $(0,0,0)$ to $(1,1,1)$.

A sketch of the proof is the following: Let $T = T_1 \cup T_2 \cup T_3$. Observe that $S = T_1 \cap T_2 \cap T_3$. It can be proved the following facts: (1) $V_Q(B) = 0$ for every 3-box B in \mathbb{I}^3 such that $B \cap T$ is, at most, a finite set of points; (2) $V_Q(B)$ is equal to the area of $B \cap T$ divided by $\sqrt{2}$ for every 3-box B in \mathbb{I}^3 such that $B \cap T$ is a surface contained in one of the triangles T_1, T_2, T_3 ; (3) every 3-box B in \mathbb{I}^3 such that $B \cap T$ contains a subsegment of S can be decomposed as union of 3-boxes of the types considered in (1) and (2) and a 3-box of type $[a, b]^3$, and $V_Q([a, b]^3) = (b - a)(b - 2a)$; (4) the area of $T \cap [a, b]^3$ is equal to $3(b - a)^2/\sqrt{2}$, whence the mass spread on $(T \setminus S) \cap [a, b]^3$ is $3(b - a)^2/2$, and the

mass spread on $S \cap [a, b]^3$ (the line segment connecting the point (a,a,a) to (b,b,b)) is $-(b^2 - a^2)/2$; (5) from (2) and (4) we conclude that Q assigns mass $1/2$ uniformly to each of the triangles T_1, T_2, T_3 and mass $-1/2$ (not uniformly) on the line segment S .

Thus, there exists a stochastic signed measure μ_Q such that $\mu_Q(B) = V_Q(B)$ for every 3-box B in \mathbb{I}^3 : the difference between the measure μ_Q^+ obtained by spreading mass $1/2$ uniformly to each of the triangles T_1, T_2, T_3 , and the measure μ_Q^- obtained by spreading mass $1/2$ on the segment S by reversing the sign of the mass spreading by Q on S .

However, not every proper n -quasi-copula induces a stochastic signed measure on \mathbb{I}^n . Next we prove this fact by looking at the mass distribution of the n -quasi-copula W^n , the pointwise best possible lower bound for both sets of n -copulas and n -quasi-copulas (recall (1.1)).

Before proceeding, we must introduce some notation. For any integer $k \geq 2$, let $T_k = \{1, 2, \dots, k\}$. We partition \mathbb{I}^n into k^n n -boxes, namely:

$$R(i_1, i_2, \dots, i_n) = \prod_{j=1}^n \left[\frac{i_j - 1}{k}, \frac{i_j}{k} \right] \quad (i_1, i_2, \dots, i_n) \in T_k^n. \tag{2.1}$$

For our purpose, we have to compute the W^n -volumes of these k^n n -boxes. In order to make these computations, we need the next two technical lemmas. Recall that, for every $x \in \mathbb{R}$ and $z \in \mathbb{Z}$, the binomial coefficient is defined by

$$\binom{x}{z} = \begin{cases} \frac{x(x-1)\cdots(x-z+1)}{z!}, & z > 0, \\ 1, & z = 0, \\ 0, & z < 0. \end{cases}$$

The following lemma recalls some easy combinatorial identities (see Eqs. (1.4), (1.25) and (1.13) in [18], respectively).

Lemma 2.1. *Let $n \geq 0$ and $p \geq 1$ be integers, and let x be a real number. Then:*

$$\sum_{j=0}^n (-1)^j \binom{x}{j} = (-1)^n \binom{x-1}{n}, \tag{2.2}$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0, \end{cases} \tag{2.3}$$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} j^p = \begin{cases} 0, & n > p, \\ (-1)^n n!, & n = p. \end{cases} \tag{2.4}$$

Observe that identity (2.3) is an immediate consequence of (2.2).

Lemma 2.2. *Let α and β be real numbers, and let n and m be integers such that $0 < m \leq n$. Then*

$$\sum_{j=0}^m (-1)^j \binom{n}{j} (\alpha + j\beta) = (-1)^m \left[\binom{n-1}{m} \alpha + \binom{n-2}{m-1} n\beta \right] = (-1)^m \binom{n-2}{m-1} \left[\frac{n-1}{m} \alpha + n\beta \right]. \tag{2.5}$$

Moreover, the first equality also holds when $0 = m \leq n$.

Proof. We consider the case $0 < m < n$: the proofs of the other cases ($0 < m = n$ and $0 = m \leq n$) are simple. We prove the first equality in (2.5): the second one is immediate. Taking $n=m$ and $x=n$ in (2.2) we have

$$\sum_{j=0}^m (-1)^j \binom{n}{j} = (-1)^m \binom{n-1}{m}. \tag{2.6}$$

On the other hand, it is easy to check the following equalities (the last one follows from identity (2.2)):

$$\sum_{j=0}^m (-1)^j \binom{n}{j} j = n \sum_{j=1}^m (-1)^j \binom{n-1}{j-1} = -n \sum_{i=0}^{m-1} (-1)^i \binom{n-1}{i} = (-1)^m \binom{n-2}{m-1} n. \tag{2.7}$$

So, from (2.6) and (2.7), we have

$$\begin{aligned} \sum_{j=0}^m (-1)^j \binom{n}{j} (\alpha + j\beta) &= \alpha \sum_{j=0}^m (-1)^j \binom{n}{j} + \beta \sum_{j=0}^m (-1)^j \binom{n}{j} j \\ &= (-1)^m \left[\binom{n-1}{m} \alpha + \binom{n-2}{m-1} n\beta \right], \end{aligned}$$

which completes the proof. \square

The following result gives the W^n -volumes of the n -boxes $R(i_1, i_2, \dots, i_n)$ defined by (2.1).

Theorem 2.1. *Let $n \geq 2$ and $k \geq 1$ be integers. Then, for every $(i_1, i_2, \dots, i_n) \in T_k^n$ we have*

$$V_{W^n}(R(i_1, i_2, \dots, i_n)) = \begin{cases} \frac{(-1)^{m-1}}{k} \binom{n-2}{m-1}, & 1 \leq m \leq \min(k, n-1), \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

where m is the integer given by

$$m = m(i_1, i_2, \dots, i_n) = \sum_{s=1}^n i_s - k(n-1). \tag{2.9}$$

Proof. Observe that the possible values for m are $(1-k)n+k, 1+(1-k)n+k, \dots, k$, where $(1-k)n+k \leq 0$ since $k \geq 2$. From the definition of W^n -volume, we have

$$\begin{aligned} V_{W^n}(R(i_1, i_2, \dots, i_n)) &= \max\left(\frac{1}{k} \sum_{s=1}^n i_s - n + 1, 0\right) - \binom{n}{1} \max\left(\frac{1}{k} \left(\sum_{s=1}^n i_s - 1\right) - n + 1, 0\right) \\ &\quad + \binom{n}{2} \max\left(\frac{1}{k} \left(\sum_{s=1}^n i_s - 2\right) - n + 1, 0\right) \\ &\quad - \dots + (-1)^n \binom{n}{n} \max\left(\frac{1}{k} \left(\sum_{s=1}^n i_s - n\right) - n + 1, 0\right) \\ &= \sum_{j=0}^n (-1)^j \binom{n}{j} \max\left(\frac{1}{k} \left(\sum_{s=1}^n i_s - j\right) - n + 1, 0\right). \end{aligned}$$

If $(1/k) \sum_{s=1}^n i_s - n + 1 \leq 0$, i.e., if $m \leq 0$, then it is immediate that $V_{W^n}(R(i_1, i_2, \dots, i_n))=0$. If $(1/k) \sum_{s=1}^n i_s - n - n + 1 \geq 0$, i.e., if $m \geq n$ (since $m \leq k$, this is the case $m > n - 1 = \min(k, n - 1)$), then identities (2.3) and (2.4) yield

$$\begin{aligned} V_{W^n}(R(i_1, i_2, \dots, i_n)) &= \sum_{j=0}^n (-1)^j \binom{n}{j} \left(\frac{1}{k} \sum_{s=1}^n i_s - n + 1 - \frac{j}{k}\right) \\ &= \left(\frac{1}{k} \sum_{s=1}^n i_s - n + 1\right) \sum_{j=0}^n (-1)^j \binom{n}{j} - \frac{1}{k} \sum_{j=0}^n (-1)^j \binom{n}{j} j = 0. \end{aligned}$$

Finally, we study the case $1 \leq m \leq \min(k, n - 1)$. Observe that $(1/k)(\sum_{s=1}^n i_s - j) - n + 1 \geq 0$ if, and only if, $j \leq m$. Therefore, Lemma 2.2 yields

$$\begin{aligned} V_{W^n}(R(i_1, i_2, \dots, i_n)) &= \sum_{j=0}^m (-1)^j \binom{n}{j} \left(\frac{1}{k} \sum_{s=1}^n i_s - n + 1 - \frac{j}{k} \right) = (-1)^m \binom{n-2}{m-1} \left(\frac{(n-1)m}{mk} - \frac{n}{k} \right) \\ &= \frac{(-1)^{m-1}}{k} \binom{n-2}{m-1}, \end{aligned}$$

which completes the proof. \square

In order to compute the number of n -boxes $R(i_1, i_2, \dots, i_n)$ whose W^n -volume is a given nonzero value—recall (2.8)—we need the following two lemmas. The first one rewrites a result in [3, p. 79].

Lemma 2.3. *Let n, k and z be positive integers such that $n \leq z \leq kn$. The number of solutions of the equation*

$$\sum_{s=1}^n i_s = z, \tag{2.10}$$

in positive integers i_s not exceeding k is

$$N(n, k, z) = \sum_{r=0}^{\lfloor (z-1)/k \rfloor} (-1)^r \binom{n}{r} \binom{z - kr - 1}{n - 1},$$

where $\lfloor (z - 1)/k \rfloor$ represents the integer part of the number $(z - 1)/k$.

The following result completes Lemma 2.3 for the case $z > k(n - 1)$.

Lemma 2.4. *Under the hypotheses of Lemma 2.3, if $k(n - 1) < z \leq kn$, then*

$$N(n, k, z) = \binom{(1+k)n - z - 1}{n - 1}.$$

Proof. Since $k(n - 1) + 1 \leq z < kn + 1$ then $n - 1 \leq (z - 1)/k < n$, i.e., $\lfloor (z - 1)/k \rfloor = n - 1$. Let q be an integer. Then, by considering separately q and $z - j$ in the product $\prod_{j=1}^{n-1} (q + z - j)$, we obtain

$$\binom{q + z - 1}{n - 1} = \frac{1}{(n - 1)!} \prod_{j=1}^{n-1} (q + z - j) = \frac{1}{(n - 1)!} \sum_{t=0}^{n-1} \left[q^{n-1-t} \sum_{1 \leq j_1 < \dots < j_t \leq n-1} \prod_{i=1}^t (z - j_i) \right], \tag{2.11}$$

where $\sum_{1 \leq j_1 < \dots < j_t \leq n-1} \prod_{i=1}^t (z - j_i) = 1$. Now, from Lemma 2.3 and identity (2.11), we have

$$\begin{aligned} N(n, k, z) &= \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} \binom{-kr + z - 1}{n - 1} \\ &= \frac{1}{(n - 1)!} \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} \sum_{t=0}^{n-1} \left[(-kr)^{n-1-t} \sum_{1 \leq j_1 < \dots < j_t \leq n-1} \prod_{i=1}^t (z - j_i) \right] \\ &= \frac{1}{(n - 1)!} \sum_{t=0}^{n-1} \left[\left(\sum_{r=0}^{n-1} (-1)^r \binom{n}{r} r^{n-1-t} \right) (-k)^{n-1-t} \sum_{1 \leq j_1 < \dots < j_t \leq n-1} \prod_{i=1}^t (z - j_i) \right]. \end{aligned}$$

From (2.4) we have

$$\sum_{r=0}^{n-1} (-1)^r \binom{n}{r} r^{n-1-t} = \sum_{r=0}^n (-1)^r \binom{n}{r} r^{n-1-t} - (-1)^n n^{n-1-t} = (-1)^{n-1} n^{n-1-t},$$

whence, by using the second equality in (2.11), we obtain

$$\begin{aligned} N(n, k, z) &= \frac{(-1)^{n-1}}{(n-1)!} \sum_{t=0}^{n-1} \left[(-kn)^{n-1-t} \sum_{1 \leq j_1 < \dots < j_t \leq n-1} \prod_{i=1}^t (z - j_i) \right] \\ &= \frac{(-1)^{n-1}}{(n-1)!} \prod_{j=1}^{n-1} (-kn + z - j) = \frac{1}{(n-1)!} \prod_{j=1}^{n-1} (kn - z + j). \end{aligned}$$

Finally, by taking $i = n - j$, we have

$$\frac{1}{(n-1)!} \prod_{j=1}^{n-1} (kn - z + j) = \frac{1}{(n-1)!} \prod_{i=1}^{n-1} ((1+k)n - z - i) = \binom{(1+k)n - z - 1}{n-1},$$

which completes the proof. \square

Now we can prove the following result.

Theorem 2.2. *Let n, k and m be three integers such that $1 \leq m \leq \min(k, n - 1)$, and consider the set*

$$A_{n,k,m} = \left\{ (i_1, i_2, \dots, i_n) \in T_k^n : V_{W^n}(R(i_1, i_2, \dots, i_n)) = \frac{(-1)^{m-1}}{k} \binom{n-2}{m-1} \right\}, \tag{2.12}$$

where $R(i_1, i_2, \dots, i_n)$ is the n -box given by (2.1). Then, the cardinal number of $A_{n,k,m}$ is

$$\text{Card}(A_{n,k,m}) = \binom{k - m + n - 1}{n - 1}.$$

Proof. From Theorem 2.1, $\text{Card}(A_{n,k,m})$ is the number of solutions of the equation $\sum_{s=1}^n i_s = k(n - 1) + m$, i.e., $N(n, k, k(n - 1) + m)$ (following the notation of Lemma 2.3). Since $1 \leq m \leq k$, we have that $k(n - 1) < k(n - 1) + m \leq kn$. Thus, from Lemma 2.4, the result follows. \square

As a consequence of the previous results, we can state the following theorem, in which we denote by λ_n the Lebesgue measure in \mathbb{R}^n .

Theorem 2.3. *Let n be an integer such that $n \geq 3$. For any positive real number M , there exists a finite set of n -boxes $\{J_1, J_2, \dots, J_p\}$ in \mathbb{I}^n whose interiors are pairwise disjoint and satisfying*

$$\sum_{i=1}^p V_{W^n}(J_i) > M \quad \text{and} \quad \sum_{i=1}^p \lambda_n(J_i) < \frac{1}{M}.$$

And, similarly, there exists a finite set of n -boxes $\{J'_1, J'_2, \dots, J'_q\}$ in \mathbb{I}^n with pairwise disjoint interiors and satisfying

$$\sum_{i=1}^q V_{W^n}(J'_i) < -M \quad \text{and} \quad \sum_{i=1}^q \lambda_n(J'_i) < \frac{1}{M}.$$

Proof. Let k be any integer such that $k \geq n - 1$. We consider the n -boxes $R(i_1, i_2, \dots, i_n)$ such that $1 \leq m = \sum_{s=1}^n i_s - k(n - 1) \leq \min(k, n - 1) = n - 1$ (recall (2.9)). For every k and m satisfying these conditions, we consider

the set of indices $A_{n,k,m}$ given by (2.12). Then

$$\begin{aligned} \sum_{(i_1, i_2, \dots, i_n) \in A_{n,k,m}} V_{W^n}(R(i_1, i_2, \dots, i_n)) &= \frac{(-1)^{m-1}}{k} \binom{n-2}{m-1} \binom{k-m+n-1}{n-1} \\ &= \frac{(-1)^{m-1}}{k} \binom{n-2}{m-1} \frac{\prod_{i=1}^{n-1} (k-m+i)}{(n-1)!} \end{aligned} \tag{2.13}$$

and

$$\sum_{(i_1, i_2, \dots, i_n) \in A_{n,k,m}} \lambda_n(R(i_1, i_2, \dots, i_n)) = \frac{1}{k^n} \binom{k-m+n-1}{n-1} = \frac{\prod_{i=1}^{n-1} (k-m+i)}{(n-1)!k^n}. \tag{2.14}$$

Observe that expression (2.13) is a polynomial function in k whose highest degree term is

$$\frac{(-1)^{m-1}}{(n-1)!} \binom{n-2}{m-1} k^{n-2}.$$

Thus, if we fix the integer m and let k tend to ∞ , then

$$\lim_{k \rightarrow \infty} (-1)^{m-1} \sum_{(i_1, i_2, \dots, i_n) \in A_{n,k,m}} V_{W^n}(R(i_1, i_2, \dots, i_n)) = \begin{cases} \infty & \text{if } m \text{ is odd,} \\ -\infty & \text{if } m \text{ is even.} \end{cases}$$

On the other hand, the expression (2.14) is a rational function in k which tends to 0 as k tends to ∞ , whence the result follows immediately. \square

Observe that Theorem 2.3 does not hold for $n=2$: in this case the only possible value for m is $m=1$, $A_{2,k,1} = \{(i_1, i_2) \in T_k^2 : V_{W^2}(R(i_1, i_2)) = 1/k\}$ and $\sum_{(i_1, i_2) \in A_{2,k,1}} V_{W^2}(R(i_1, i_2)) = 1$.

As a consequence of Theorem 2.3, for $n \geq 3$ not every proper n -quasi-copula Q induces a stochastic signed measure μ_Q on \mathbb{I}^n (such that, of course, $\mu_Q(\mathbb{I}^k \times A \times \mathbb{I}^{n-k-1}) = \lambda_1(A)$ for every Lebesgue measurable set A in \mathbb{I} and every $k = 0, 1, 2, \dots, n-1$), as stated in the following theorem.

Theorem 2.4. *For every $n \geq 3$, the n -quasi-copula W^n does not induce a stochastic signed measure on \mathbb{I}^n , i.e., there exists no stochastic signed measure μ on \mathbb{I}^n such that $\mu(B) = V_{W^n}(B)$ for every n -box B in \mathbb{I}^n , and $\mu(\mathbb{I}^k \times A \times \mathbb{I}^{n-k-1}) = \lambda_1(A)$ for every Lebesgue measurable set A in \mathbb{I} and every $k = 0, 1, 2, \dots, n-1$.*

Proof. Suppose there exists a stochastic signed measure μ such that $\mu(B) = V_{W^n}(B)$ for every n -box B in \mathbb{I}^n . Let $D_+, D_- \subset \mathbb{I}^n$ be such that (D_+, D_-) is a Hahn decomposition of \mathbb{I}^n with respect to μ . Since $\mu(D_+) + \mu(D_-) = \mu(\mathbb{I}^n) = \lambda_1(\mathbb{I}) = 1$, then $-\infty < \mu(D_-) \leq 0 < \mu(D_+) < \infty$. Theorem 2.3 assures that there exists a Borel set E in \mathbb{I}^n such that $\mu(D_+) < \mu(E)$. So we have the following string of equalities and inequalities: $\mu(E \cap D_+) \leq \mu(E \cap D_+) + \mu(E^c \cap D_+) = \mu(D_+) < \mu(E) = \mu(E \cap D_+) + \mu(E \cap D_-) \leq \mu(E \cap D_+)$, whence we have a contradiction, and this completes the proof. \square

Finally, note that the limit of a sequence of proper n -quasi-copulas that induce a stochastic signed measure on \mathbb{I}^n can be a proper n -quasi-copula that does not induce a signed measure on \mathbb{I}^n (see [33]).

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