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Kendall distribution functions and associative copulas

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Abstract

In this paper, we prove that any Kendall distribution function is the Kendall distribution function of some associative copula. We use this result to show that each equivalence class of the relation "to have the same Kendall distribution function as" defined on the set of copulas contains a unique associative representative.
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1. Introduction

If (X_1, Y_1) and (X_2, Y_2) are two continuous random vectors with respective distribution functions H_1 and H_2, and K_i denotes the distribution function of the random variable $H_i(X_i, Y_i)$, $i = 1, 2$, the *Kendall stochastic ordering* $\prec_K$ was defined by Capéraà et al. [3] as follows:

$$(X_1, Y_1) \prec_K (X_2, Y_2) \quad \text{if and only if } K_1(t) \geqslant K_2(t) \text{ for all } t \in \mathbb{R}. \tag{1}$$

Kendall's name is associated with this ordering since, for every continuous random pair (X, Y) with distribution function H, Genest and Rivest [12] have proved that the population version of the measure of association known as *Kendall's tau* can be expressed as

$$\tau(X, Y) = 3 - 4\int_0^1 K(t)\,\mathrm{d}t,$$

where K is the distribution function of $H(X, Y)$. We [20] have called this function as the *Kendall distribution function of* (X, Y), and we have studied several properties of these functions. For instance, we have used the Bertino family of copulas (see the work of Fredricks and Nelsen [10]) to show that every distribution function K satisfying that $K(0^-) = 0$

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and $K(t) \geqslant t$ ($t \in [0, 1]$) is the Kendall distribution function of some pair of random variables; and we have also shown that each equivalence class of the relation "to have the same Kendall distribution function as" defined on the set of copulas contains a unique Bertino copula (for applications, see the work of Fountain et al. [9] and also our previous work [19]).

We now prove that any Kendall distribution function is the Kendall distribution function of some associative copula. The main interest is that it permits to show that each equivalence class of the relation "to have the same Kendall distribution function as" contains a unique associative copula.

2. Preliminaries

A (bivariate) copula is the restriction to $[0, 1]^2$ of a continuous bivariate distribution function whose margins are uniform on [0, 1]. The importance of copulas is described in *Sklar's Theorem* [21]: Let *X* and *Y* be two random variables with joint distribution function *H* and marginal distribution functions *F* and *G*, respectively. Then, there exists a copula *C* (which is uniquely determined on Range $F \times$ Range G) such that $H(x, y) = C(F(x), G(y))$ for all x, y. If *F* and *G* are continuous, then *C* is unique. Thus, copulas link joint distribution functions to their one-dimensional margins. For a complete survey on copulas, see the monograph of Nelsen [18].

As a consequence of Sklar's Theorem, the Kendall distribution function of a continuous random pair (X, Y) depends only on the copula *C* of *X* and *Y*. Thus, we have

$$K(t) = \Pr[C(U, V) \leqslant t], \tag{2}$$

where (U, V) is any pair with copula *C* and margins uniform on [0, 1]. In view of Eq. (1), Kendall distribution functions induce an equivalence relation $\equiv_K$ on the set of copulas: If C_1 and C_2 are two copulas with respective Kendall distribution functions K_1 and K_2, then

$$C_1 \equiv_K C_2 \quad \text{if and only if } K_1(t) = K_2(t) \text{ for all } t \in [0, 1]. \tag{3}$$

Let *M* and *W* denote the Fréchet–Hoeffding upper and lower bound copulas, respectively, which for any copula *C* satisfy the following inequalities: $\max(u + v - 1, 0) = W(u, v) \leqslant C(u, v) \leqslant M(u, v) = \min(u, v)$ for every (u, v) in $[0, 1]^2$.

A copula *C* is *associative* if $C(C(u, v), w) = C(u, C(v, w))$ for every u, v, w in [0, 1]. It is known that every associative copula is a (continuous) *triangular norm* (briefly, *t-norm*); and every t-norm which satisfies the 1-Lipschitz property is a copula (see the monographs of Alsina et al. [1] and Klement et al. [14]). Triangular norms are used in fuzzy set theory in a wide range of applications (see, for instance, the monographs of Klement and Mesiar [13] and Klement et al. [14], and the paper of Calvo et al. [2]); and copulas in fields such as preference modelling (see the works of De Baets and De Meyer [4], De Baets and Fodor [5] and Díaz et al. [6–8]). Klement et al. [15] have shown that the class of associative copulas is compact. Finally, Mayor et al. [16] have studied the structure of associative discrete copulas.

Let Φ denote the set of functions $\varphi: [0, 1] \longrightarrow [0, \infty]$ which are continuous, strictly decreasing, convex, and such that $\varphi(1) = 0$. Each $\varphi \in \Phi$ has a *pseudo-inverse* $\varphi^{[-1]}: [0, \infty] \longrightarrow [0, 1]$ given by $\varphi^{[-1]}(t) = \varphi^{-1}(t)$ if $t \in [0, \infty]$, and $\varphi^{[-1]}(t) = 0$ if $t \in [\varphi(0), \infty]$. $\varphi^{[-1]}$ is also continuous, strictly decreasing on $[0, \varphi(0)]$, and $\varphi^{[-1]}(0) = 1$. If $\varphi(0) = \infty$, then we have $\varphi^{[-1]}(t) = \varphi^{-1}(t)$ for all $t \in [0, \infty]$. Each member of Φ generates a copula *C* given by $C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$, (u, v) in $[0, 1]^2$. These copulas are called *Archimedean*, and φ is the *generator* of *C*. Note that Archimedean copulas are associative. Genest and MacKay [11] and Nelsen [18] have studied other properties of Archimedean copulas.

We need to recall the definition of an ordinal sum of copulas. Let $P = \{J_i\}$ denote a countable system of closed (and possibly degenerate) intervals $J_i = [a_i, b_i]$ whose union is [0, 1] and whose pairwise intersections contain, at most, a common endpoint. Let $\{C_i\}$ be a collection of copulas with the same indexing as $\{J_i\}$ such that $C_i = M$ for all indices i for which $J_i = \{a_i\}$, i.e., $a_i = b_i$. Let $\Delta(P) = \bigcup\{J_i \times J_i : J_i \in P \text{ and } a_i < b_i\}$. Then, the *ordinal sum* of the collection $\{C_i\}$ with respect to the set *P* is the copula *C* defined by

$$C(u, v) = \begin{cases} a_i + (b_i - a_i) \cdot C_i\left(\dfrac{u - a_i}{b_i - a_i}, \dfrac{v - a_i}{b_i - a_i}\right) & \text{if } (u, v) \in J_i \times J_i \subset \Delta(P), \\ M(u, v) & \text{otherwise.} \end{cases} \tag{4}$$

We will use some notation. The class of the sets $P = \{J_i\}$ as in the definition of ordinal sum will be denoted by $\mathcal{P}$. Let f be a real function defined on $[a, b]$ (or on a dense subset of $[a, b]$ including a and b) having only removable or jump discontinuities. Then $l^- f$ and $l^+ f$ will denote the functions defined on $[a, b]$ by $l^- f(x) = f(x^-)$ and $l^+ f(x) = f(x^+)$, where $f(x^-)$ (respectively, $f(x^+)$) denotes the left-hand limit (respectively, right-hand limit) of f at x (if it exists). It is easy to prove that $l^- f$ is left continuous on $[a, b]$ and $l^+ f$ is right continuous, whence $l^- l^- f = l^- f = l^- l^+ f$ and $l^+ l^+ f = l^+ f = l^+ l^- f$. Finally, if G is a function and $\{G_n\}$ is a sequence of functions, then $G_n \to G$ will denote point-wise convergence to G.

3. The results

We begin this section with two lemmas. The second lemma is due to Genest and Rivest [12].

Lemma 1. *Let K be a right-continuous distribution function such that* $K(0^-) = 0$ *and, for every t in* $[0, 1]$, $K(t) \geqslant t$. *Then there exists a set* $P = \{[a_i, b_i]\}$ *in* $\mathcal{P}$ *satisfying the following properties*:

(i) $K(b_i{}^-) = b_i$ *for all i*,
(ii) *for each i such that* $a_i < b_i$, *either* (a) $K(t^-) > t$ *for all* $t \in (a_i, b_i)$, *or* (b) $K(t) = t$ *for all* $t \in [a_i, b_i)$,
(iii) *for each i such that* $a_i = b_i = c$, *we have that* $K(c) = c$.

Proof. We know that the function $l^- K$ is left-continuous and non-decreasing. If $l^- K(t) = t$ for all t in $(0, 1)$, there is nothing to be proved. Suppose that there exist points t in $(0, 1)$ such that $l^- K(t) > t$, and let t_0 be one of those points. Since $l^- K$ is left continuous, we have that $\lim_{t \to t_0^-} (l^- K(t) - t) = l^- K(t_0) - t_0 > 0$. Therefore, if we take ε in $(0, l^- K(t_0) - t_0)$, then we can find a number $\delta > 0$ such that $l^- K(t) - t$ is in $(l^- K(t_0) - t_0 - \varepsilon, l^- K(t_0) - t_0 + \varepsilon)$ whenever $t \in (t_0 - \delta, t_0]$. In particular, $l^- K(t) - t > l^- K(t_0) - t_0 - \varepsilon > 0$, and then $l^- K(t) > t$ for every t in $(t_0 - \delta, t_0]$. Since $l^- K$ is non-decreasing, we have that $l^- K(t) \geqslant l^- K(t_0) > t$ whenever $t \in (t_0, l^- K(t_0))$. The last two inequalities yield the following result: for every $t_0 \in (0, 1)$ such that $l^- K(t_0) > t_0$, we can find a number $\gamma > 0$ such that $l^- K(t) > t$ for every t in $(t_0 - \gamma, t_0 + \gamma)$. For each $t_0 \in (0, 1)$ such that $l^- K(t_0) > t_0$, let $\Delta(t_0)$ be the class constituted by the open intervals $I \subset (0, 1)$ such that $t_0 \in I$ and $l^- K(t) > t$ for every $t \in I$. We just know that $\Delta(t_0) \neq \emptyset$. Let $I(t_0) = \bigcup \{I : I \in \Delta(t_0)\}$. It is clear that $I(t_0)$ is an open interval containing t_0, i.e., $I(t_0)$ is the maximal element of $\Delta(t_0)$. Moreover, if $I(t_0) = (a, b)$, we have that $l^- K(b) = b$; otherwise, if $l^- K(b) > b$, we would be able to find a number $\gamma > 0$ such that $l^- K(t) > t$ for every t in $(b - \gamma, b + \gamma)$, and then $(a, b + \gamma)$ would belong to $\Delta(t_0)$, contrary to the maximality of (a, b).

Now, we consider the class $I(K)$ constituted by the intervals $[a, b] \subset (0, 1]$ such that (a, b) is the maximal element of a class $\Delta(t_0)$, for some t_0 in $(0, 1)$ satisfying that $l^- K(t_0) > t_0$. The condition of maximal element yields that the intervals of the class $I(K)$ overlaps, at most, on their endpoints; and, in its turn, this fact implies that the class $I(K)$ is countable. If U_K denotes the union of the intervals in the class $I(K)$, it is clear that $[0, 1] \backslash U_K$ is a countable disjoint union of both open intervals and closed degenerate (singletons) intervals. The closures of these intervals together with the intervals of the class $I(K)$ form a set P which belongs to $\mathcal{P}$. Let $J_i = [a_i, b_i]$ be an element of P with $a_i < b_i$. If $J_i \in I(K)$, then we have that $K(b_i{}^-) = l^- K(b_i) = b_i$ and $K(t^-) = l^- K(t) > t$ for all t in (a_i, b_i). If $J_i \notin I(K)$ is a non-degenerate interval, then we have that $l^- K(t) = t$ for all t in J_i. In particular, $K(b_i{}^-) = b_i$, and $K(t) = l^+ K(t) = l^+ l^- K(t) = l^+ t = t$ for each t in $[a_i, b_i)$.

Finally, let $J_i = [c, c]$ one of the degenerate intervals of P. Then, there exists an infinite sequence of non-degenerate intervals $[a_i, b_i]$ of P such that $K(t) = t$ for all $t \in [a_i, b_i]$ whose left endpoints constitute a decreasing sequence of real numbers $\{x_n\}$ converging to c. Thus, $K(x_n) = x_n$ implies that $K(c) = c$. On the other hand, we can also find an infinite sequence of non-degenerate intervals of P whose endpoints constitute an increasing sequence of real numbers $\{x_n\}$ converging to c. Since $l^- K$ is left-continuous, we have that $\{l^- K(x_n)\} \to l^- K(c)$; moreover, $l^- K(x_n) = x_n$, and then $l^- K(c) = c$. This completes the proof. $\square$

Lemma 2. *Let K be a distribution function such that* $K(0^-) = 0$ *and* $K(t^-) > t$ *for all t in* $(0, 1)$. *Then there exists a unique Archimedean copula C such that the Kendall distribution function of C is K. The generator of C is given by*

$$\varphi_C(x) = \exp\left(\int_{x_0}^{x} \frac{\mathrm{d}t}{t - K(t)} \right), \tag{5}$$

where x_0 *is a constant arbitrarily chosen in* $(0, 1)$.

We are now in position to prove the following result, in which we extend the result in Lemma 2 without the restriction "$K(t^-) > t$ for all t in (0, 1)."

Theorem 3. *Let K be a right-continuous distribution function such that $K(0^-) = 0$ and $K(t) \geqslant t$ for all t in* [0, 1]. *Then, there exists an associative copula C such that the distribution function of $T = C(U, V)$ is K, where U and V are random variables uniformly distributed on* [0, 1] *with copula C.*

Proof. Let $P \in \mathcal{P}$ be the set associated with the function K given by Lemma 1. For each J_i such that $a_i < b_i$, let K_i be the right-continuous distribution function given by

$$K_i(t) = \begin{cases} 0 & \text{if } t < 0, \\ \dfrac{K(a_i + (b_i - a_i)t) - a_i}{b_i - a_i} & \text{if } 0 \leqslant t \leqslant 1, \\ 1 & \text{if } t > 1. \end{cases}$$

It follows that $K_i(0^-) = 0$, and either

(a) if $K(t^-) > t$ for all t in (a_i, b_i), then

$$K_i(t^-) = \frac{l^- K(a_i + (b_i - a_i)t) - a_i}{b_i - a_i} > \frac{a_i + (b_i - a_i)t - a_i}{b_i - a_i} = t$$

for all t in (0, 1), or

(b) if $K(t) = t$ for all t in $[a_i, b_i)$, then

$$K_i(t) = \frac{K(a_i + (b_i - a_i)t) - a_i}{b_i - a_i} = \frac{a_i + (b_i - a_i)t - a_i}{b_i - a_i} = t$$

for all t in [0, 1).

For each i such that $a_i < b_i$, we construct a copula C_i as follows: If (a) holds, let C_i be the Archimedean copula constructed via Lemma 2 from the Kendall distribution function K_i; if (b) holds, let $C_i = M$. Note that if the joint distribution function of two random variables U and V is C_i, then the distribution function of $C_i(U, V)$ is K_i. Now, let C be the ordinal sum of the collection $\{C_i\}$ with respect to the set P—recall the expression given in Eq. (4). We claim that the distribution function of $T = C(U, V)$ is K. Let $t \in J_i$, with $a_i < b_i$. Then, the uniformity of both U and V and the structure of C as an ordinal sum produce that

$$\Pr[T \leqslant t] = \Pr[T \leqslant a_i] + \Pr[a_i < T \leqslant t] = a_i + \iint_{S_i} \mathrm{d}C(u, v),$$

where $S_i = \{(u, v) \in J_i \times J_i : C(u, v) \leqslant t\}$. But for (u, v) in S_i, we have that

$$C(u, v) = a_i + (b_i - a_i) \cdot C_i\left(\frac{u - a_i}{b_i - a_i}, \frac{v - a_i}{b_i - a_i}\right)$$

and

$$\mathrm{d}C(u, v) = (b_i - a_i)\,\mathrm{d}C_i\left(\frac{u - a_i}{b_i - a_i}, \frac{v - a_i}{b_i - a_i}\right),$$

so that $\Pr[T \leqslant t]$ can be rewritten as

$$\Pr[T \leqslant t] = a_i + (b_i - a_i) \iint_{S_i} \mathrm{d}C_i\left(\frac{u - a_i}{b_i - a_i}, \frac{v - a_i}{b_i - a_i}\right).$$

Let $s = (u - a_i)/(b_i - a_i)$ and $w = (v - a_i)/(b_i - a_i)$. Then

$$\Pr[T \leqslant t] = a_i + (b_i - a_i) \iint_{R_i} \mathrm{d}C_i(s, w),$$

where $R_i = \{(s, w) \in [0, 1]^2 : C_i(s, w) \leqslant (t - a_i)/(b_i - a_i)\}$. If $C_i = M$, then

$$\Pr[T \leqslant t] = a_i + (b_i - a_i) \cdot \frac{t - a_i}{b_i - a_i} = t = K(t).$$

If C_i is the Archimedean copula such that the distribution function of $C_i(U, V)$ is K_i, then

$$\Pr[T \leqslant t] = a_i + (b_i - a_i) \cdot K_i\left(\frac{t - a_i}{b_i - a_i}\right) = a_i + (b_i - a_i) \cdot \frac{K(t) - a_i}{b_i - a_i} = K(t).$$

Finally, let $t \in J_i$, with $a_i = b_i$. Then, it is clear that $\Pr[T \leqslant t] = t$; and, from Lemma 1, we know that $K(t) = t$, which completes the proof. □

It is well known that a copula is associative if, and only if, it is an ordinal sum of a collection of copulas $\{C_i\}$ such that, for each i, C_i is either Archimedean or equal to M (see, for instance, the monograph of Alsina el al. [1]). This fact and the proof of Theorem 3 lead us to the following result.

Theorem 4. *Under the same hypotheses of Theorem 3, there exists a unique associative copula C such that the distribution function of T is K. As a consequence, every equivalence class given by the equivalence relation in Eq. (3) has one and only one associative element, which is the copula given by Theorem 3.*

Proof. Let C be a copula, and let $\overline{C}$ be the equivalence class of C according to the relation $\equiv_K$ defined by Eq. (3). Let U and V be two random variables whose joint distribution function is C. If we consider the function K_C defined by Eq. (2) and construct the copula D associated with K_C given by Theorem 3, then we obtain that $D \in \overline{C}$. Suppose that E is another associative copula in $\overline{C}$. Then E must be an ordinal sum of a collection of copulas $\{E_i\}$ with respect to a set $\{[c_i, d_i]\} \in \mathcal{P}$ such that, for each i, E_i is either Archimedean or equal to M. Let U' and V' be two random variables whose joint distribution function is E. Then, as in the proof of Theorem 3, we can obtain that

$$\Pr[E(U', V') \leqslant t] = c_i + (d_i - c_i) \iint_{R_i} \mathrm{d}E_i(s, w)$$

for all $t \in [c_i, d_i]$ such that $c_i < d_i$, where $R_i = \{(s, w) \in [0, 1]^2 : E_i(s, w) \leqslant (t - c_i)/(d_i - c_i)\}$. If $E_i = M$, again $\Pr[E(U', V') \leqslant t] = t$. If E_i is an Archimedean copula, then

$$\Pr[E(U', V') \leqslant t] = c_i + (d_i - c_i) \cdot K_{E_i}\left(\frac{t - c_i}{d_i - c_i}\right) > t.$$

Therefore, since $\Pr[E(U', V') \leqslant t] = K_C(t)$, the sets $\{[a_i, b_i]\}$ and $\{[c_i, d_i]\}$ must coincide. And it must also occur that $D_i = E_i$ for all i since there is an one-to-one relation between Archimedean copulas and the functions K such that $K(t^-) > t$ in $(0, 1)$—recall Lemma 2—which completes the proof. □

We [20] have shown that the equivalence classes of $\equiv_K$ containing the copulas M and W for the Fréchet–Hoeffding bounds are singletons. In the following example, we provide a non singleton equivalence class, which—in view of Theorem 4—contains an associative element.

Example 1. Let C_1 be the copula whose probability mass is spread uniformly on the two segments in $[0, 1]^2$ joining the points $(0, 1/2)$ to $(1/2, 1)$, and $(1/2, 0)$ to $(1, 1/2)$; and C_2 the copula whose probability mass is spread uniformly on the segments joining the points $(0, 0)$, $(1/2, 1)$ and $(1, 0)$. The copulas C_1—which is a *shuffle of Min* (see the work of Mikusiński et al. [17])—and C_2 are given by

$$C_1(u, v) = \max(0, u + v - 1, \min(u, v - 1/2), \min(u - 1/2, v))$$

and

$$C_2(u, v) = \min(u, \max(v/2, u + v - 1))$$

for every (u, v) in $[0, 1]^2$, respectively. It is easy to show that $K_{C_1}(t) = K_{C_2}(t) = \min(2t, 1)$ for all $t \in [0, 1]$. C_1 and C_2 are not associative (for instance, note that $C_1(C_1(1/4, 3/4), 3/4) = 1/4 > 0 = C_1(1/4, C_1(3/4, 3/4))$ and

$C_2(C_2(3/4, 1/2), 1/2) = 1/4 > 1/8 = C_2(3/4, C_2(1/2, 1/2))$. Thus, in view of Theorem 4, the equivalence class has an associative element: The Archimedean copula C defined in Lemma 2, and whose generator is given by Eq. (5). Taking $x_0 = 1/2$ in Eq. (5), we easily obtain that

$$\varphi_C(x) = \begin{cases} 1/(2x) & \text{if } x \in [0, 1/2) \\ 2(1-x) & \text{if } x \in [1/2, 1] \end{cases} \text{ and } \varphi_C^{-1}(y) = \begin{cases} 1 - y/2 & \text{if } y \in [0, 1], \\ 1/(2y) & \text{if } y \in (1, \infty]. \end{cases}$$

Whence, after some elementary algebra, C is given by

$$C(u, v) = \begin{cases} \dfrac{uv}{u+v} & \text{if } (u, v) \in [0, 1/2]^2, \\ \max\left(\dfrac{1}{4(2-u-v)}, u+v-1\right) & \text{if } (u, v) \in [1/2, 1]^2, \\ \dfrac{\min(u, v)}{1 + 4\min(u, v)[1 - \max(u, v)]} & \text{otherwise.} \end{cases}$$

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