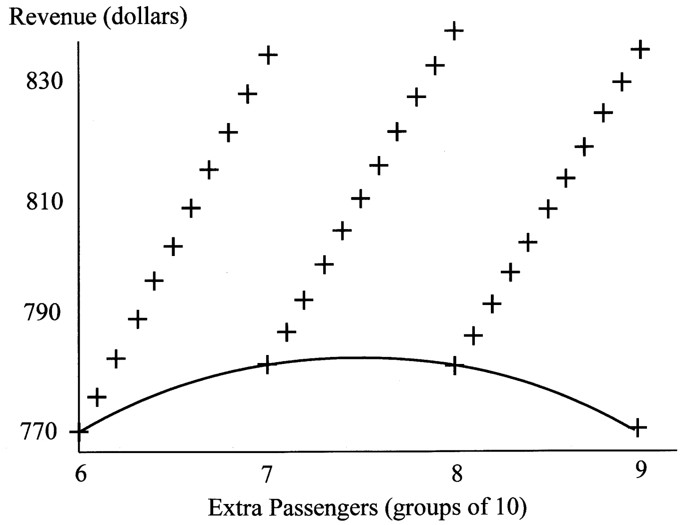

be presented in algebra courses, using the vertex formula for a parabola. However, the interpretation of the numerical results requires students to think about the actual situation and relate their "mathematical" solutions to the "real world" situation.

An Improved Remainder Estimate for Use with the Integral Test

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Nearly every modern calculus text contains the following result in the chapter on infinite series: If $\sum_{i=0}^{\infty} f(i)$ converges to S by the integral test, and $S_n = \sum_{i=1}^{n} f(i)$ denotes the nth partial sum of the series, then the "remainder" $R_n = S - S_n = \sum_{i=n+1}^{\infty} f(i)$ satisfies

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx. \qquad (1)$$

The hypotheses for the integral test require that f be continuous, positive, and decreasing on $[1, \infty)$. In [**1**], R. K. Morley showed that if, as is often the case, f is also *convex* (concave up) on $[1, \infty)$, then the "traditional" estimate (1) can be improved to

$$\int_{n}^{\infty} f(x)\,dx - \frac{1}{2}f(n) \le R_n \le \int_{n}^{\infty} f(x)\,dx - \frac{1}{2}f(n+1). \qquad (2)$$

The purpose of this Capsule is to note that, under the same hypotheses, these estimates can be further sharpened to

$$\frac{1}{2}f(n+1) + \int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n+1/2}^{\infty} f(x)\,dx. \qquad (3)$$

The proof of (3) follows directly from the observation that for convex functions, the midpoint rule underestimates integrals while the trapezoidal rule overestimates integrals. (This observation holds for convergent improper integrals as well as for proper integrals.) The right-hand inequality in (3) follows from the fact that $R_n = \sum_{i=n+1}^{\infty} f(i)$ is a midpoint rule estimate for $\int_{n+1/2}^{\infty} f(x)\,dx$. Similarly, the left-hand inequality in (3) follows from the fact that $\sum_{i=n+1}^{\infty} f(i) - \frac{1}{2} f(n+1)$ is a trapezoidal rule estimate for $\int_{n+1}^{\infty} f(x)\,dx$.

In actual practice, when we use (1)–(3) in estimating S, we replace R_n by $S - S_n$ and solve for S. Thus, these inequalities become

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_{n}^{\infty} f(x)\,dx, \tag{4}$$

$$S_n + \int_{n}^{\infty} f(x)\,dx - \frac{1}{2} f(n) \le S \le S_n + \int_{n}^{\infty} f(x)\,dx - \frac{1}{2} f(n+1) \tag{5}$$

$$S_n + \frac{1}{2} f(n+1) + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_{n+1/2}^{\infty} f(x)\,dx. \tag{6}$$

For example, consider estimating the sum S of $\sum_{i=1}^{\infty} i^{-4}$ using $S_5 \cong 1.080352$ (all calculations have been rounded to six places). Since $f(x) = x^{-4}$ is convex on $[1, \infty)$, inequalities (4)–(6) yield the following intervals:

Table 1. Using S_5 to approximate $\sum_{i=1}^{\infty} i^{-4}$

Method	Interval
(1)	$1.081895 \le S \le 1.083019$
(2)	$1.082219 \le S \le 1.082633$
(3)	$1.082281 \le S \le 1.082355$

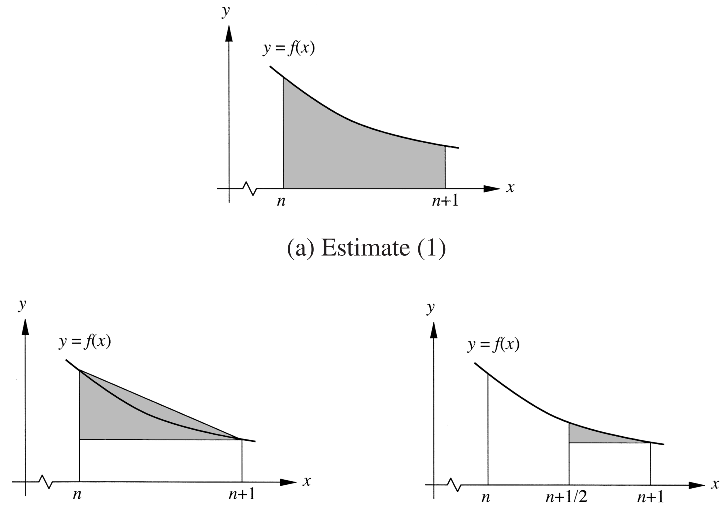
Of course, the sum S is actually equal to $\pi^4/90 \cong 1.082323$.

In this example, the third interval is less than 7% as wide as the first and less than 18% as wide as the second (.000074 versus .001123 and .000414, respectively), a substantial improvement in precision at virtually no "cost" in additional computation. In general, the width of the interval for R_n in (1) is given by $\int_n^{n+1} f(x)\,dx$, the width in (2) is $\frac{1}{2}[f(n) - f(n+1)]$, while the width of the interval in (3) is $\int_{n+1/2}^{n+1} f(x)\,dx - \frac{1}{2} f(n+1)$. The improvement in precision is illustrated in Figure 1, where the areas of the shaded regions are numerically equal to the interval widths in each case.

The geometric interpretation of the interval width in Figure 1(c) leads to an efficient method to determine the number n of terms to use in (3) to obtain an approximation to any desired precision ε. The area of the shaded region in Figure 1(c) is no larger than the area of the triangle with the same vertices, $\frac{1}{4}[f(n+1/2) - f(n+1)]$. If we wish the interval width to be less than ε, then n must satisfy $f(n+1/2) - f(n+1) < 4\varepsilon$. When f is differentiable on $(1, \infty)$, we can apply the mean value theorem to f on $(n+1/2, n+1)$ to obtain $f(n+1/2) - f(n+1) = f'(c)(-1/2)$ for some c in $(n+1/2, n+1)$. But f is convex, so f' is increasing. Thus, $-f'$ is decreasing, and so $-f'(c) \le -f'(n+1/2)$. Hence, $f(n+1/2) - f(n+1) < 4\varepsilon$ whenever

$$-f'(n+1/2) < 8\varepsilon. \tag{7}$$

Returning to our example, let us find the number n of terms necessary to approximate $\sum_{i=1}^{\infty} i^{-4}$ correctly to six decimal places using inequalities (6). Solving (7) with



(a) Estimate (1)

(b) Estimate (2)

(c) Estimate (3)

Figure 1.

$f(x) = x^{-4}$ and $\varepsilon = .5 \times 10^{-6}$ yields $n \geq 16$. Using $S_{16} \cong 1.08224917$ (to 8 places) and (3) yields $1.08232300 \leq S \leq 1.08232337$; hence $S \cong 1.082323$, correct to 6 places. To obtain the same accuracy with (2) requires $n \geq 21$, and with (1) requires $n \geq 38$.

Reference

1. R. K. Morley, The remainder in computing by series, *American Mathematical Monthly* **57** (1950), 550–551.

A Modified Discrete SIR Model

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A couple of years ago in a Calculus I course, I introduced my students to the classic Kermack-McKendrick SIR model [**1**] for the spread of an epidemic:

$$\begin{aligned} \frac{dS}{dt} &= -aS(t)I(t), \\ \frac{dI}{dt} &= aS(t)I(t) - bI(t), \\ \frac{dR}{dt} &= bI(t), \end{aligned} \tag{8}$$

where $S(t)$, $I(t)$, and $R(t)$ represent, respectively, the number of "susceptibles," "infecteds," and "recovereds" in a population at time t, with time measured in days. The