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Journal of Multivariate Analysis 90 (2004) 348–358
Journal of Multivariate Analysis
http://www.elsevier.com/locate/jmva

Best-possible bounds on sets of bivariate distribution functions

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Received 4 September 2001

Abstract

The fundamental best-possible bounds inequality for bivariate distribution functions with given margins is the Fréchet–Hoeffding inequality: If H denotes the joint distribution function of random variables X and Y whose margins are F and G, respectively, then $\max(0, F(x) + G(y) - 1) \leqslant H(x, y) \leqslant \min(F(x), G(y))$ for all x, y in $[-\infty, \infty]$. In this paper we employ copulas and quasi-copulas to find similar best-possible bounds on arbitrary sets of bivariate distribution functions with given margins. As an application, we discuss bounds for a bivariate distribution function H with given margins F and G when the values of H are known at quartiles of X and Y.

AMS 2000 subject classifications: primary 60E15; secondary 62E10; 62H20

Keywords: Bounds; Copulas; Distribution functions; Kendall's tau; Quartiles; Quasi-copulas

1. Introduction

The fundamental best-possible bounds inequality for bivariate distribution functions with given margins was obtained by Hoeffding [7] and Fréchet [2] independently some 50–60 years ago: let X and Y be random variables with

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doi:10.1016/j.jmva.2003.09.002

distribution functions F and G, respectively. If H denotes the joint distribution function of X and Y, then

$$\max(0, F(x) + G(y) - 1) \leqslant H(x,y) \leqslant \min(F(x), G(y)) \tag{1.1}$$

for all x, y in $\mathbf{R} = [-\infty, \infty]$. Furthermore, the bounds in (1.1) are themselves bivariate distribution functions with margins F and G.

In an earlier paper [10], the authors found bounds on the set of joint distribution functions of continuous random variables with known margins and a known value of a measure of association such as Kendall's tau or Spearman's rho. In this paper we present a method for finding bounds on arbitrary sets of joint distribution functions of continuous random variables with known margins, and illustrate the procedure by finding bounds when the values of the joint distribution function are known at the quartiles of the marginal distributions.

As is often the case in dealing with multivariate distribution functions, the use of copulas simplifies matters. The importance of copulas in statistical modeling is described in Sklar's theorem [11]: Let X and Y be random variables with joint distribution function H and marginal distribution functions F and G, respectively. Then there exists a copula C (which is uniquely determined on Range $F \times$ Range G) such that $H(x,y) = C(F(x), G(y))$ for all x, y in $\mathbf{R}$. Thus copulas link joint distribution functions to their one-dimensional margins. For further details, see [8].

With copulas and Sklar's theorem, the Fréchet–Hoeffding inequality (1.1) becomes

$$W(u,v) = \max(0, u + v - 1) \leqslant C(u,v) \leqslant \min(u,v) = M(u,v). \tag{1.2}$$

for all u, v in $\mathbf{I}$. The Fréchet–Hoeffding bounds M and W in (1.2) are themselves copulas.

The Fréchet–Hoeffding bounds in (1.1) can often be narrowed when we possess additional information about H, as the following example illustrates.

Example 1.1. Let X, Y, F, G, and H be as in the first paragraph, and suppose that the value θ of H is known at a point in $\mathbf{R}^2$ whose coordinates are medians $\tilde{x}$ and $\tilde{y}$ of X and Y, respectively, i.e., $H(\tilde{x}, \tilde{y}) = \theta$. Since $F(\tilde{x}) = 1/2$ and $G(\tilde{y}) = 1/2$, we have $H(\tilde{x}, \tilde{y}) = C(F(\tilde{x}), G(\tilde{y})) = C(1/2, 1/2) = \theta$. Then (see [8, Theorem 3.2.2]) the bounds on H are given by $\underline{C}_\theta(F(x), G(y)) \leqslant H(x,y) \leqslant \bar{C}_\theta(F(x), G(y))$ for all x, y in $\mathbf{R}$, where $\underline{C}_\theta$ and $\bar{C}_\theta$ are the copulas given by

$$\underline{C}_\theta(u,v) = \max\{W(u,v), \theta - (1/2 - u)^+ - (1/2 - v)^+\}$$

and

$$\bar{C}_\theta(u,v) = \min\{M(u,v), \theta + (u - 1/2)^+ + (v - 1/2)^+\},$$

where $x^+ = \max(x, 0)$.

Since our methods for finding bounds on arbitrary sets of distribution functions with given margins involve quasi-copulas as well as copulas, we review some elementary properties of quasi-copulas before proceeding (see [6] for more details). A (two-dimensional) *quasi-copula* is a function $Q\colon \mathbf{I}^2 \to \mathbf{I}$ which satisfies the same boundary conditions as does a copula,

$$Q(t,0) = Q(0,t) = 0 \text{ and } Q(t,1) = Q(1,t) = t \text{ for every } t \text{ in } \mathbf{I}, \tag{1.3}$$

but in place of the 2-*increasing* condition for a copula C, i.e.,

$$V_C([u_1,u_2] \times [v_1,v_2]) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geqslant 0 \tag{1.4}$$

for all $u_1,\ u_2,\ v_1,\ v_2$ in $\mathbf{I}$ such that $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$ (V_C is called the *C-volume* of the rectangle $[u_1,u_2] \times [v_1,v_2]$), the weaker conditions that Q is nondecreasing in each variable, i.e.,

$$Q(s_1,t) \leqslant Q(s_2,t) \text{ and } Q(t,s_1) \leqslant Q(t,s_2)$$
$$\text{for every } s_1,\ s_2, t \text{ in } \mathbf{I} \text{ with } s_1 \leqslant s_2; \tag{1.5}$$

and the Lipschitz condition,

$$|Q(u_2,v_2) - Q(u_1,v_1)| \leqslant |u_2 - u_1| + |v_2 - v_1| \text{ for all } u_1,u_2,v_1,v_2 \text{ in } \mathbf{I}. \tag{1.6}$$

Conditions (1.5) and (1.6) together are equivalent to requiring that (1.4) holds whenever at least one of $u_1,\ u_2,\ v_1,\ v_2$ is equal to 0 or to 1. While every copula is a quasi-copula, there exist *proper* quasi-copulas, i.e., quasi-copulas which are not copulas. As with copulas, every quasi-copula satisfies the Fréchet–Hoeffding inequality (1.2).

Quasi-copulas first arose in the process of characterizing, within a certain class of operations on distribution functions, those which derive from the corresponding operations on random variables [1,9]. In the next section, we use quasi-copulas to express the pointwise best-possible bounds on nonempty sets of distribution functions, copulas or quasi-copulas; and in the following section we present an application to sets of copulas (or distribution functions) with some common property such as a common diagonal section or common values at quartiles.

2. The bounds

Definition 2.1. Let $\mathbf{S}$ be a nonempty set of bivariate functions with a common domain D. Then $\underline{S}$ and $\bar{S}$ denote, respectively, the pointwise infimum and supremum of $\mathbf{S}$, i.e., for each (u,v) in D,

$$\underline{S}(u,v) = \inf\{S(u,v) | S \in \mathbf{S}\} \text{ and } \bar{S}(u,v) = \sup\{S(u,v) | S \in \mathbf{S}\}. \tag{2.1}$$

$\underline{S}$ and $\bar{S}$ are bounds for $\mathbf{S}$ since for each S in $\mathbf{S}$, $\underline{S} \leqslant S \leqslant \bar{S}$ on D, and are clearly pointwise best-possible. In general, however, neither $\underline{S}$ nor $\bar{S}$ is an element of $\mathbf{S}$. In the sequel we consider cases in which $\mathbf{S}$ is a set of bivariate distribution functions, a set of copulas, or a set of quasi-copulas.

In the following theorem, we show that the bounds in (2.1) for a nonempty set of quasi-copulas are also quasi-copulas.

Theorem 2.2. *Let* $\mathbf{Q}$ *be a nonempty set of quasi-copulas. Then* $\underline{Q}$ *and* $\bar{Q}$ *are quasi-copulas.*

Proof. We prove that $\bar{Q}$ is a quasi-copula, the proof for $\underline{Q}$ is similar. For the boundary conditions (1.3), we have $\bar{Q}(u,0) = \sup\{Q(u,0)|Q \in \mathbf{Q}\} = \sup\{0|Q \in \mathbf{Q}\} = 0$ and $\bar{Q}(u,1) = \sup\{Q(u,1)|Q \in \mathbf{Q}\} = \sup\{u|Q \in \mathbf{Q}\} = u$, and similarly $\bar{Q}(0,v) = 0$ and $\bar{Q}(1,v) = v$. Since each quasi-copula is nondecreasing in its arguments, we have $\bar{Q}(u_1,v) = \sup\{Q(u_1,v)|Q \in \mathbf{Q}\} \leqslant \sup\{Q(u_2,v)|Q \in \mathbf{Q}\} = \bar{Q}(u_2,v)$, so that $\bar{Q}$ is nondecreasing in u (and similarly in v).

To show that $\bar{Q}$ is Lipschitz, it will suffice to show that whenever $u_1 \leqslant u_2$, $\bar{Q}(u_2,v) - \bar{Q}(u_1,v) \leqslant u_2 - u_1$. Let u_1, u_2 be fixed in $\mathbf{I}$ with $u_1 \leqslant u_2$. For any $\varepsilon > 0$, there exists a quasi-copula Q_ε such that $Q_\varepsilon(u_2,v) > \bar{Q}(u_2,v) - \varepsilon$. Since $Q_\varepsilon(u_1,v) \leqslant \bar{Q}(u_1,v)$, it follows that $\bar{Q}(u_2,v) - \bar{Q}(u_1,v) < Q_\varepsilon(u_2,v) + \varepsilon - Q_\varepsilon(u_1,v) \leqslant u_2 - u_1 + \varepsilon$. Since this is true for every $\varepsilon > 0$, we have $\bar{Q}(u_2,v) - \bar{Q}(u_1,v) \leqslant u_2 - u_1$, as required.

Since every copula is a quasi-copula, the following corollary is immediate:

Corollary 2.3. *Let* $\mathbf{C}$ *be a nonempty set of copulas. Then* $\underline{C}$ *and* $\bar{C}$ *are quasi-copulas.*

The corollary cannot be strengthened to conclude that $\underline{C}$ and $\bar{C}$ are copulas, as the following example illustrates.

Example 2.1. For any θ in $\mathbf{I}$, let C_θ be the function given by

$$C_\theta(u,v) = \begin{cases} \min(u, v-\theta), & (u,v) \in [0, 1-\theta] \times [\theta, 1], \\ \min(u+\theta-1, v), & (u,v) \in [1-\theta, 1] \times [0, \theta], \\ W(u,v) & \text{otherwise.} \end{cases}$$

Each C_θ is a copula [8, Exercise 3.9], and if U and V are uniform $(0,1)$ random variables whose joint distribution function is C_θ, then $V = U \oplus \theta$ with probability one, where $\oplus$ denotes addition mod 1. The support of C_θ is illustrated in Fig. 1. If we let $\mathbf{C}$ be the set $\{C_{1/3}, C_{2/3}\}$, then $\bar{C}$ is the quasi-copula given by

$$\bar{C}(u,v) = \begin{cases} \max(0, u-1/3, v-1/3, u+v-1), & -1/3 \leqslant v-u \leqslant 1/3, \\ M(u,v) & \text{otherwise.} \end{cases} \tag{2.2}$$

$\bar{C}$ is not a copula, since $V_{\bar{C}}([1/3,2/3]^2) = -1/3$, which violates (1.4). Using [8, Example 3.4], it is easy to construct an analogous example for which $\underline{C}$ is a proper quasi-copula.

The next result, whose proof is immediate, is central to our purpose since it shows that in order to study bounds on sets of joint distribution functions (with common margins), we need only study bounds on the corresponding sets of copulas.

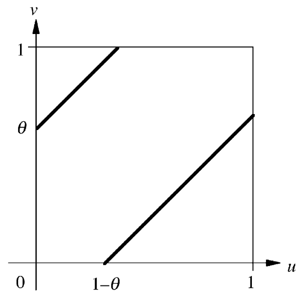

Fig. 1. The support of the copula in Example 2.1.

Theorem 2.4. *Let F and G be continuous (one-dimensional) distribution functions. Let $\mathbf{H}$ be a nonempty set of bivariate distribution functions with the property that if H is in $\mathbf{H}$, then the margins of H are F and G, i.e., $H(x,\infty) = F(x)$ and $H(\infty, y) = G(y)$. Let $\mathbf{C}$ denote the set of copulas corresponding to the elements of $\mathbf{H}$, i.e.,*

$$\mathbf{C} = \{C | C \text{ is a copula, and for some } H \in \mathbf{H},$$
$$H(x,y) = C(F(x), G(y)) \text{ for all } (x,y) \in \mathbf{R}^2\}.$$

Then for all (x,y) in $\mathbf{R}^2$, $\underline{H}(x,y) = \underline{C}(F(x), G(y))$ and $\bar{H}(x,y) = \bar{C}(F(x), G(y))$.

The bounds $\underline{H}$ and $\bar{H}$ in Theorem 2.4 are "quasi-distribution functions," as the margins F and G are linked by a quasi-copula rather than a copula, as in Sklar's theorem.

3. Copulas with given diagonal sections

The *diagonal section* δ_C of a copula C is the function given by $\delta_C(t) = C(t,t)$ for t in $\mathbf{I}$ (and similarly for a quasi-copula). When U and V are uniform $(0,1)$ random variables whose joint distribution function is C, then δ_C is the distribution function of $\max(U,V)$. In Example 1.1 we found the bounds on a bivariate distribution function H when the value of H was known at medians of X and Y. Further suppose that H is known at quartiles of X and Y, i.e., for $i = 1,2,3$; suppose x_i, y_i satisfy $F(x_i) = i/4 = G(y_i)$, and $H(x_i, y_i) = \theta_i$. In terms of the copula C of X and Y, we have $C(i/4, i/4) = \theta_i$ for $i = 1,2,3$; that is, the value of the copula is known at three points on its diagonal section. Given this information we can find bounds on the copula (and hence, via Theorem 2.4, on the joint distribution function) of X and Y. But first we investigate bounds on sets of copulas and quasi-copulas with a common diagonal section.

Before proceeding, we need several definitions. A *diagonal* is a function $\delta : \mathbf{I} \to \mathbf{I}$ with the properties (i) $\delta(1) = 1$; (ii) $\delta(t) \leqslant t$ for all t in $\mathbf{I}$, and (iii) $0 \leqslant \delta(t_2) - \delta(t_1) \leqslant 2(t_2 - t_1)$ for all t_1, t_2 in $\mathbf{I}$ such that $t_1 \leqslant t_2$. Note that the diagonal section of any quasi-copula (and thus any copula) is a diagonal, and for any diagonal δ, there exist copulas (and thus quasi-copulas) whose diagonal section is δ. For example (see [3,4]), the function K_δ with domain $\mathbf{I}^2$ given by

$$K_\delta(u,v) = \min\{u, v, (1/2)[\delta(u) + \delta(v)]\}, \tag{3.1}$$

is a symmetric (i.e. $K_\delta(u,v) = K_\delta(v,u)$ for all u, v in $\mathbf{I}$) copula with diagonal section δ. Copulas in this family characterize identically distributed continuous random variables X, Y for which the joint distribution of $\max(X, Y)$ and $\min(X, Y)$ is the Fréchet–Hoeffding upper bound [4].

For any diagonal δ, let $\mathbf{C}_\delta$ be the set of copulas C whose diagonal section δ_C is δ, i.e.,

$$\mathbf{C}_\delta = \{C | C \text{ is a copula, } C(t,t) = \delta(t) \text{ for all } t \in \mathbf{I}\}. \tag{3.2}$$

Analogously, we define

$$\mathbf{Q}_\delta = \{Q | Q \text{ is a quasi-copula, } Q(t,t) = \delta(t) \text{ for all } t \in \mathbf{I}\}. \tag{3.3}$$

Note that for any δ, $\mathbf{C}_\delta \subseteq \mathbf{Q}_\delta$, and each set is nonempty since K_δ from (3.1) is a copula.

For any diagonal δ we now define two functions, each with domain $\mathbf{I}^2$:

$$A_\delta(u,v) = \begin{cases} \min\{u, v - \max(t - \delta(t) | t \in [u,v])\}, & u \leqslant v, \\ \min\{v, u - \max(t - \delta(t) | t \in [v,u])\}, & v \leqslant u, \end{cases} \tag{3.4}$$

and

$$B_\delta(u,v) = \begin{cases} u - \min(t - \delta(t) | t \in [u,v]), & u \leqslant v, \\ v - \min(t - \delta(t) | t \in [v,u]), & v \leqslant u. \end{cases} \tag{3.5}$$

The following lemma presents the basic properties of A_δ and B_δ:

Lemma 3.1. *Let δ be a diagonal and let $\mathbf{C}_\delta$, $\mathbf{Q}_\delta$, A_δ and B_δ be given by* (3.2), (3.3),(3.4) *and* (3.5) *respectively. Then* (a) *A_δ and B_δ are symmetric*, (b) *$A_\delta \in \mathbf{Q}_\delta$, and* (c) *$B_\delta \in \mathbf{C}_\delta$.*

Proof. We prove only (b), since the proof of (a) is immediate and the proof of (c) can be found in [5], where one can also find a statistical characterization of random variables with copula B_δ. For the boundary conditions (1.3), since $0 \leqslant t - \delta(t) \leqslant \min(1-t, t)$ for all t in $\mathbf{I}$, $\max(t - \delta(t) | t \in [0,u]) \leqslant \max(t | t \in [0,u]) = u$ and $\max(t - \delta(t) | t \in [u,1]) \leqslant \max(1 - t | t \in [u,1]) = 1 - u$; so that $A_\delta(u,0) = \min\{0, u - \max(t - \delta(t) | t \in [0,u])\} = 0$ and $A_\delta(u,1) = \min\{u, 1 - \max(t - \delta(t) | t \in [u,1])\} = u$, and similarly $A_\delta(0,v) = 0$ and $A_\delta(1,v) = v$ since A_δ is symmetric.

To prove that $A_\delta(u_1,v) \leqslant A_\delta(u_2,v)$ for every u_1, u_2, v in $\mathbf{I}$ with $u_1 \leqslant u_2$ (and similarly $A_\delta(v,u_1) \leqslant A_\delta(v,u_2)$ by symmetry), suppose that $u_1 \leqslant v \leqslant u_2$ (the cases $u_1 \leqslant u_2 \leqslant v$ and $v \leqslant u_1 \leqslant u_2$ can be proved by similar arguments), let $t_0 \in [v, u_2]$ such that $\max(t - \delta(t) | t \in [v,u_2]) = t_0 - \delta(t_0)$, then $\max(t - \delta(t) | t \in [v,u_2]) - \max(t - \delta(t) | t \in [u_1,v]) \leqslant \max(t - \delta(t) | t \in [v,u_2]) - \max(t - \delta(t) | t \in [v,v]) = t_0 - \delta(t_0) - v + \delta(v) \leqslant u_2 - \delta(v) - v + \delta(v) = u_2 - v$; so that $v - \max(t - \delta(t) | t \in [u_1,v]) \leqslant u_2 - \max(t - \delta(t) | t \in [v,u_2])$; and hence $A_\delta(u_1,v) \leqslant A_\delta(u_2,v)$.

To show that the conditions in (1.6) hold, it will suffice to show that whenever $u_1 \leqslant u_2$, $A_\delta(u_2, v) - A_\delta(u_1, v) \leqslant u_2 - u_1$. Consider again $u_1 \leqslant v \leqslant u_2$ (the cases $u_1 \leqslant u_2 \leqslant v$ and $v \leqslant u_1 \leqslant u_2$ are similar). If $A_\delta(u_1, v) = u_1$, then $A_\delta(u_2, v) - A_\delta(u_1, v) = \min\{v, u_2 - \max(t - \delta(t)|t \in [v, u_2])\} - u_1 \leqslant u_2 - u_1$. Suppose now that $A_\delta(u_1, v) = v - \max(t - \delta(t)|t \in [u_1, v])$, and let $t_1 \in [u_1, v]$ such that $t_1 - \delta(t_1) = \max(t - \delta(t)|t \in [u_1, v])$. Noting that $\delta(v) - \delta(t_1) \leqslant 2(v - t_1)$, we have

$$\begin{aligned} A_\delta(u_2, v) - A_\delta(u_1, v) &\leqslant u_2 - \max(t - \delta(t)|t \in [v, u_2]) - v + \max(t - \delta(t)|t \in [u_1, v]), \\ &\leqslant u_2 - \max(t - \delta(t)|t \in [v, v]) - v + t_1 - \delta(t_1), \\ &= u_2 - v + \delta(v) - v + t_1 - \delta(t_1), \\ &\leqslant u_2 - 2v + t_1 + 2(v - t_1) = u_2 - t_1 \leqslant u_2 - u_1. \end{aligned}$$

Finally observe that $A_\delta(t, t) = \delta(t)$, which completes the proof. □

The result in part (b) cannot be strengthened to conclude that A_δ is a copula, as the following example illustrates.

Example 3.1. Let $\delta(t) = \max(0, t - 1/3, 2t - 1)$ (see Fig. 2(a)). Using (3.4) to construct A_δ, we obtain the proper quasi-copula given by $\bar{C}$ in (2.2).

The next lemma shows that A_δ and B_δ are bounds for the sets $\mathbf{C}_\delta$ and $\mathbf{Q}_\delta$.

Lemma 3.2. *Let δ, $\mathbf{C}_\delta$, $\mathbf{Q}_\delta$, A_δ, and B_δ be as in Lemma* 3.1. *Then* (a) *for any Q in $\mathbf{Q}_\delta$, $B_\delta \leqslant Q \leqslant A_\delta$, and* (b) *for any C in $\mathbf{C}_\delta$, $B_\delta \leqslant C \leqslant A_\delta$.*

Proof. We prove only (a), since (b) then follows from the observation that $\mathbf{C}_\delta \subseteq \mathbf{Q}_\delta$. Let Q be in $\mathbf{Q}_\delta$, assume $0 \leqslant u \leqslant v \leqslant 1$, and let t be any number in $[u, v]$. Since Q is nondecreasing in each argument, $Q(t, v) \geqslant Q(t, t)$, or $Q(t, v) \geqslant \delta(t)$. Since Q is Lipschitz, $Q(t, v) - Q(u, v) \leqslant t - u$, or $Q(u, v) \geqslant u - t + Q(t, v)$. Hence $Q(u, v) \geqslant u - t + \delta(t)$ for all t in $[u, v]$, so that $Q(u, v) \geqslant u - \min\{t - \delta(t)|t \in [u, v]\}$. Since an analogous result holds for $0 \leqslant v \leqslant u \leqslant 1$, $Q \geqslant B_\delta$.

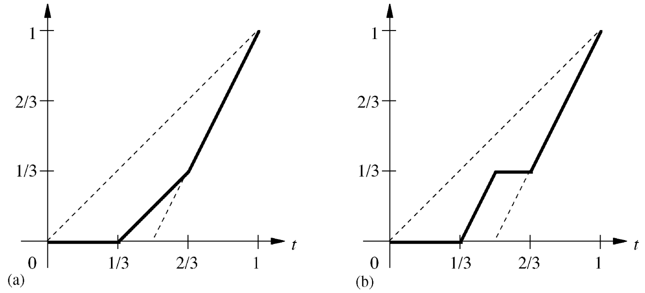

Fig. 2. The diagonals in (a) Example 3.1 and (b) Example 3.2.

Similarly (again for $0 \leqslant u \leqslant v \leqslant 1$ and any t in $[u, v]$), $Q(t, v) - Q(t, t) \leqslant v - t$, or $Q(t, v) \leqslant v - t + \delta(t)$. But $Q(u, v) \leqslant Q(t, v)$, and hence $Q(u, v) \leqslant v - t + \delta(t)$ for all t in $[u, v]$, so that $Q(u, v) \leqslant v - \max\{t - \delta(t) | t \in [u, v]\}$. But $Q(u, v) \leqslant M(u, v) = u$ as well, and thus $Q(u, v) \leqslant \min\{u, v - \max(t - \delta(t) | t \in [u, v])\}$. Since an analogous result holds for $0 \leqslant v \leqslant u \leqslant 1, Q \leqslant A_\delta$, which completes the proof.

Theorem 3.3. *Let* δ, $\mathbf{C}_\delta$, $\mathbf{Q}_\delta$, A_δ, *and* B_δ *be as in Lemma* 3.1. *If* $\underline{C}_\delta$, $\bar{C}_\delta$, $\underline{Q}_\delta$, *and* $\bar{Q}_\delta$ *denote the pointwise infima and suprema* (2.1) *of* $\mathbf{C}_\delta$ *and* $\mathbf{Q}_\delta$, *respectively, then* (a) $\underline{C}_\delta = \underline{Q}_\delta = B_\delta$; (b) $\bar{C}_\delta \leqslant \bar{Q}_\delta = A_\delta$; *and* (c) *if* A_δ *is a copula, then* $\bar{C}_\delta = A_\delta$.

Proof. Since $B_\delta \in \mathbf{Q}_\delta$ and $B_\delta \leqslant Q$ for all $Q \in \mathbf{Q}_\delta$, $\underline{Q}_\delta = B_\delta$. Similarly $\underline{C}_\delta = B_\delta$. Since $\mathbf{C}_\delta \subseteq \mathbf{Q}_\delta$, $A_\delta \in \mathbf{Q}_\delta$ and $A_\delta \geqslant Q$ for all $Q \in \mathbf{Q}_\delta$, $\bar{C}_\delta \leqslant \bar{Q}_\delta = A_\delta$. If A_δ is a copula, then $A_\delta \leqslant \bar{C}_\delta$, hence $\bar{C}_\delta = A_\delta$. □

The next two theorems give conditions under which A_δ is a copula.

Theorem 3.4. *Let* δ *be a diagonal, and let* A_δ *and* K_δ *be given by* (3.4) *and* (3.1), *respectively. Then* A_δ *is a copula if and only if* $A_\delta = K_\delta$.

Proof. We first show that if C is any symmetric copula whose diagonal section is δ, then $C \leqslant K_\delta$. For every u, v in $\mathbf{I}$, $C(v, u) = C(u, v)$; and from (1.4), $C(v, v) - C(v, u) - C(u, v) + C(u, u) \geqslant 0$; hence $C(u, v) \leqslant (1/2)[\delta(u) + \delta(v)]$. But since $C(u, v) \leqslant M(u, v)$ as well, it follows that $C(u, v) \leqslant K_\delta(u, v)$.

Now assume A_δ is a copula. Since A_δ is symmetric, $A_\delta \leqslant K_\delta$. But since $K_\delta \in \mathbf{C}_\delta$, $K_\delta \leqslant A_\delta$, hence $A_\delta = K_\delta$. The converse is trivial. □

Theorem 3.5. *Let* δ *be a diagonal, and let* A_δ *and* K_δ *be given by* (3.4) *and* (3.1), *respectively. Then* $A_\delta = K_\delta$ *if and only if the graph of* $v = \delta(u)$ *is piecewise linear with* (a) *each segment having slope* 0, 1, *or* 2; *and* (b) *each segment having at least one of its endpoints on the line* $v = u$.

Proof. Since δ is continuous and $\delta(t) \leqslant t$ for each t in $\mathbf{I}$, there exists a countable set of intervals $\mathscr{J} = \{I_i = [a_i, b_i] | i \in \mathscr{I}\}$ such that $\bigcup\{I_i | i \in I\} = \mathbf{I}$; if $i \neq j$, each $I_i \cap I_j$ is either empty or contains a single point; and for each i in $\mathscr{I}$, $\delta(t) = t$ for t in I_i or else $\delta(t) < t$ for t in (a_i, b_i) and $\delta(a_i) = a_i$, $\delta(b_i) = b_i$. Thus we only need to prove that $A_\delta = K_\delta$ if, and only if, for the intervals in the last case, δ is given by

$$\delta(t) = \begin{cases} a_i, & a_i \leqslant t \leqslant (a_i + b_i)/2, \\ 2t - b_i, & (a_i + b_i)/2 < t \leqslant b_i. \end{cases}$$

The necessary condition in this equivalence follows from a long and technical proof (see [12] for details). Conversely, assume $v \leqslant u$ (the case $u \leqslant v$ follows since A_δ and K_δ

are symmetric). If u belongs to an interval $[a_i, b_i]$ in $\mathscr{J}$ such that $\delta(t) = t$ for t in $[a_i, b_i]$, then $K_\delta(u, v) = v$ for all v in $[0, u]$; and if u belongs to an interval $[a_i, b_i]$ in $\mathscr{J}$ such that $\delta(t) < t$ for t in (a_i, b_i), then K_δ is given by

$$K_\delta(u,v) = \begin{cases} a_i, & a_i < v \leqslant u \leqslant (a_i + b_i)/2, \\ u - (b_i - a_i)/2, & u - (b_i - a_i)/2 < v \leqslant (a_i + b_i)/2 < u < b_i, \\ u + v - b_i, & (a_i + b_i)/2 < v \leqslant u < b_i, \\ v & \text{elsewhere.} \end{cases}$$

It is now easy (but tedious, see [12] for details) to use (3.4) and hypothesis (a) to establish $A_\delta = K_\delta$, as required. □

When A_δ is a proper quasi-copula, we may have $\bar{C}_\delta \neq A_\delta$, as the following example illustrates.

Example 3.2. For the diagonal $\delta(t) = \min[\max(0, 2t - 2/3), \max(1/3, 2t - 1)]$, whose graph is piecewise linear connecting the points $(0,0)$, $(1/3, 0)$, $(1/2, 1/3)$, $(2/3, 1/3)$, and (1,1) (see Fig. 2(b)), tedious but elementary calculations yield the proper quasi-copula

$$A_\delta(u,v) = \begin{cases} \min(1/3, u + v - 2/3), & (u,v) \in [1/3, 2/3]^2, \\ \bar{C}(u,v) & \text{otherwise,} \end{cases}$$

where $\bar{C}$ is given by (2.2). Let C be any copula in $\mathbf{C}_\delta$. Now $C(1/3, 1/2) \leqslant 1/6$ since $V_C([1/3, 1] \times [1/3, 1/2]) \geqslant 0$, and $C(1/2, 2/3) \leqslant \delta(2/3) = 1/3$. But $V_C([1/3, 1/2] \times [1/2, 2/3]) \geqslant 0$ so that

$$\begin{aligned} C(1/3, 2/3) &\leqslant C(1/2, 2/3) + C(1/3, 1/2) - \delta(1/2), \\ &\leqslant 1/3 + 1/6 - 1/3 = 1/6. \end{aligned}$$

Hence $\bar{C}_\delta(1/3, 2/3) \leqslant 1/6$. However, $A_\delta(1/3, 2/3) = 1/3$, and thus $\bar{C}_\delta \neq A_\delta$.

We now return to the application with which we introduced this section. If C is a copula with $\delta_C(i/4) = \theta_i$ for $i = 0, 1, 2, 3, 4$ (with $\theta_0 = 0$ and $\theta_4 = 1$) then it is again tedious but elementary to verify that $\delta_L(t) \leqslant \delta_C(t) \leqslant \delta_U(t)$ for t in $\mathbf{I}$, where δ_L and δ_U are the diagonals given by

$$\delta_L(t) = \min\{\max(\theta_{i-1}, 2t - i/2 + \theta_i) | i = 1, 2, 3, 4\}$$

and

$$\delta_U(t) = \max\{\min(t, 2t - i/2 + \theta_i, \theta_{i+1}) | i = 0, 1, 2, 3\}.$$

It now follows from Theorem 3.3 that lower and upper bounds on the copula C of X and Y are B_{δ_L} and A_{δ_U}, respectively. The lower bound B_{δ_L} is best-possible, since B_{δ_L} is a copula with diagonal section δ_L for which $\delta_L(i/4) = \theta_i$ for $i = 1, 2, 3$. However, the upper bound A_{δ_U} is best-possible (and equal to K_{δ_U}) if and only if δ_U satisfies the conditions in Theorem 3.5, or equivalently, if and only if $\theta_1 + \theta_2 \geqslant 1/2$ and $\theta_2 + \theta_3 \geqslant 1$.

We conclude by noting we can also obtain bounds on the population version of Kendall's tau for X and Y when H is known at the quartiles of X and Y as a consequence of the following theorem:

Theorem 3.6. *Let δ be a diagonal, and let K_δ and B_δ be given by* (3.1) *and* (3.5) *respectively. Then the population versions of Kendall's tau for random variables with copulas K_δ and B_δ are given, respectively, by*

$$\tau(K_\delta) = 4\int_0^1 \delta(t)\,dt - 1 \text{ and } \tau(B_\delta) = 8\int_0^1 \delta(t)\,dt - 3.$$

The expression for $\tau(K_\delta)$ is from [3], while that for $\tau(B_\delta)$ is from [12]. If C_1 and C_2 are any two copulas such that $C_1 \leqslant C_2$, then $\tau(C_1) \leqslant \tau(C_2)$ [8]. Hence, if we let τ_L and τ_U denote the bounds for $\tau(C)$ when $C(i/4, i/4) = \theta_i$ for $i = 1, 2, 3$; then $\tau_L = \tau(B_{\delta_L})$ and $\tau_U = \tau(K_{\delta_U})$, from which it follows that for any $\theta_1, \theta_2, \theta_3$,

$$\tau_L = 4(\theta_1^2 - \theta_1\theta_2 + \theta_2^2 - \theta_2\theta_3 + \theta_3^2) + 2(\theta_1 + \theta_2 - \theta_3) - 1,$$

and when $\theta_1 + \theta_2 \geqslant 1/2$ and $\theta_2 + \theta_3 \geqslant 1$,

$$\tau_U = 1 - 4[(1/4 - \theta_1)^2 + (1/2 - \theta_2)^2 + (3/4 - \theta_3)^2].$$

Extensions of some of the results in this paper to the multivariate case are the subject of current research.

Acknowledgments

The authors acknowledge the support of a Spanish C.I.C.Y.T. grant (PB 98-1010), the Consejería de Educación y Ciencia of the Junta de Andalucía (Spain), and their respective institutions. The authors also thank two anonymous referees for their helpful comments on an earlier version of this paper.

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