The Bertino family of copulas

Gregory A. Fredricks (`fredrix@lclark.edu`) and
Roger B. Nelsen (`nelsen@lclark.edu`)
Department of Mathematical Sciences, Lewis & Clark College

Abstract.

In this paper we present some of the salient properties of the Bertino family of copulas. We describe the support set of a Bertino copula and show that every Bertino copula is singular. We characterize Bertino copulas in terms of the joint distribution of $\max(U,V)$ and $\min(U,V)$ when U and V are uniform $[0,1]$ random variables whose copula is a Bertino copula. Finally, we find necessary and sufficient conditions for a Bertino copula to be extremal.

Keywords: Copula, extremal

AMS subject classification: 60E05, 62E10, 62H20.

1. Preliminaries

A *copula* is a function $C : I^2 \to I = [0,1]$ such that for each $t \in I$

$$C(0,t) = C(t,0) = 0 \text{ and } C(1,t) = C(t,1) = t, \tag{1.1}$$

and for each $u_1 < u_2$ and $v_1 < v_2$ in I

$$C(u_2,v_2) + C(u_1,v_1) - C(u_2,v_1) - C(u_1,v_2) \geq 0. \tag{1.2}$$

(See Nelsen (1999) for an introduction to copulas.) Each copula C uniquely determines a probability measure μ_C on I^2 by defining $C(u,v)$ to be the measure of the rectangle $[0,u] \times [0,v]$. These measures are called doubly stochastic as they have the property that for each subinterval J of I, the measures of $J \times I$ and of $I \times J$ are the length of J. The *support* of a copula C is the support of μ_C, i.e., the complement of the union of all open sets of μ_C-measure zero. A copula is *singular* if its support is a set of Lebesgue measure zero. A copula C is *symmetric* if $C(u,v) = C(v,u)$ for all $u,v \in I$.

The *diagonal section* of a copula C is the function $\delta_C : I \to I$ defined by $\delta_C(t) = C(t,t)$. A *diagonal* is any function $\delta : I \to I$ for which $\delta(0) = 0,\ \delta(1) = 1,\ \delta(t) \leq t$ for all $t \in I$, and $0 \leq \delta(t_2) - \delta(t_1) \leq 2(t_2 - t_1)$ whenever $t_1 < t_2$ in I. One can easily verify that every diagonal section is a diagonal. The converse is also true — see Fredricks and Nelsen (1997).

For a subset S of I, let $\triangle_S = \{(t,t) : t \in S\}$. We denote $\triangle_I$ simply by $\triangle$, and say that $\triangle_S$ is the part of $\triangle$ corresponding to S.

The following is a special case of a definition made by Bertino (1977). For each diagonal δ, define B_δ on I^2 by

$$B_\delta(u,v) = \min(u,v) - \min_{t\in[\{u,v\}]} \hat{\delta}(t),$$

where $\hat{\delta}(t) = t - \delta(t)$ for all $t \in I$ and $[\{u,v\}]$ denotes the closed interval with endpoints u and v.

Proposition 1.1. *For each diagonal δ, the function $\hat{\delta}$ is continuous on I, has graph in the closed triangle with vertices $(0,0), (\frac{1}{2}, \frac{1}{2})$ and $(1,0)$, and satisfies*

$$|\hat{\delta}(t_2) - \hat{\delta}(t_1)| \leq |t_2 - t_1| \quad \text{for every } t_1, t_2 \text{ in } I.$$

Proof: That $\hat{\delta} \geq 0$ on I follows from $\delta(t) \leq t$ for all $t \in I$. Since the secant slopes on the graph of δ lie between 0 and 2 inclusively, the secant slopes on the graph of $\hat{\delta}$ lie between -1 and 1 inclusively. The result follows easily.

Proposition 1.2. *Each B_δ is a symmetric copula with diagonal section δ. Moreover, if C is any copula with diagonal section δ, then $B_\delta \leq C$ on I^2.*

Proof: Fix a diagonal δ and let $B = B_\delta$. It is obvious that $B(u,v) = B(v,u)$ for all $u, v \in I$ and that $\delta_B = \delta$ on I. Note that B maps I^2 into I as $\hat{\delta} \geq 0$ on I implies that $B(u,v) \leq \min(u,v) \leq 1$, and

$$B(u,v) \geq \min(u,v) - \hat{\delta}(\min(u,v)) = \delta(\min(u,v)) \geq 0.$$

B obviously satisfies the conditions in (1.1). As for (1.2) consider $R = [u_1,u_2] \times [v_1,v_2]$ with $u_1 < u_2 < v_1 < v_2$ in I. Let $J_1 = [u_1,u_2],\ J_2 = [u_2,v_1],\ J_3 = [v_1,v_2]$ and $k_i = \min_{t\in J_i}\hat{\delta}(t)$. Then

$$\begin{aligned}\mu_B(R) = x_2 - \min(k_2,k_3) + x_1 - \min(k_1,k_2) - (x_2 - k_2) \\ -(x_1 - \min(k_1,k_2,k_3)) \\ = \min(k_1,k_2,k_3) + k_2 - \min(k_1,k_2) - \min(k_2,k_3),\end{aligned}$$

so $\mu_B(R) = k_2 - \min(k_2,k_3)$, 0, or $k_2 - \min(k_1,k_2)$ depending on whether $\min(k_1,k_2,k_3)$ is k_1, k_2, or k_3, respectively. Thus $\mu_B(R) \geq 0$. Now suppose that $u < v$ in I. Choose $s \in [u,v]$ such that $\hat{\delta}(s) = \min_{t\in[u,v]}\hat{\delta}(t)$ and note that

$$\begin{aligned}\mu_B([u,v]^2) &= \delta(v) + \delta(u) - 2(u - \hat{\delta}(s)) \\ &= [(s + \hat{\delta}(s)) - (u + \hat{\delta}(u))] + [\delta(v) - \delta(s)] \geq 0,\end{aligned}$$

as $t + \hat{\delta}(t)$ and δ are both nondecreasing on I. This establishes that B is a copula.

Suppose now that C is a copula with diagonal section δ. Fix $u < v$ in I. Then $B(u,v) \le C(u,v)$ as for each fixed $s \in [u,v]$ we have

$$\begin{aligned} u - \hat{\delta}(s) &= \delta(s) - (s-u) = \mu_C([0,s]^2) - \mu_C([u,s] \times I) \\ &\le C(u,s) \le C(u,v). \end{aligned}$$

A similar argument in the case $v < u$ establishes that $B \le C$ on I^2.

B_δ is the *Bertino copula* associated with the diagonal δ.

Example 1.3. If $\delta(t) = t$ for all $t \in I$, then $\hat{\delta} \equiv 0$ on I, so B_δ is the copula M defined to be the minimum of the arguments.

Example 1.4. If $\delta(t) = 0$ if $t \in [0, \frac{1}{2}]$ and $2t - 1$ if $t \in [\frac{1}{2}, 1]$, then $\hat{\delta}(t) = t$ if $t \in [0, \frac{1}{2}]$ and $1 - t$ if $t \in [\frac{1}{2}, 1]$. Hence

$$B_\delta(u,v) = \begin{cases} 0 & \text{if } u + v \le 1 \\ u + v - 1 & \text{if } u + v \ge 1, \end{cases}$$

which is the copula commonly denoted by W.

Example 1.5. If $\delta(t) = t^2$ for all $t \in I$, then $\hat{\delta}(t) = t - t^2$ for all $t \in I$ and

$$B_\delta(u,v) = \begin{cases} [\min(u,v)]^2 & \text{if } u + v \le 1 \\ \min(u,v) - \hat{\delta}(\max(u,v)) & \text{if } u + v \ge 1. \end{cases}$$

The support of B_δ is the union of the two diagonals of the unit square.

Example 1.6. Let $\delta = \delta_{\alpha,\beta}$ denote the diagonal whose graph consists of the line segments connecting $(0,0)$ to (α,β) and (α,β) to $(1,1)$. Specifically, assume that $\max(2\alpha - 1, 0) \le \beta < \alpha$, and note that

$$\hat{\delta}(t) = \begin{cases} \frac{\alpha-\beta}{\alpha} t & \text{if } t \in [0,\alpha] \\ \frac{\alpha-\beta}{1-\alpha}(1-t) & \text{if } t \in [\alpha, 1]. \end{cases}$$

The values of $B_\delta(u,v)$ are $u - \hat{\delta}(u)$, $u - \hat{\delta}(v)$, $v - \hat{\delta}(v)$ and $v - \hat{\delta}(u)$ on the triangles S, T, S' and T', respectively, where, in addition to (α,α), S has vertices $(0,0)$ and $(0,1)$, T has vertices $(0,1)$ and $(1,1)$, S' has vertices $(0,0)$ and $(1,0)$, and T' has vertices $(1,0)$ and $(1,1)$. The support of B_δ lies in the union of $\triangle$ and the two line segments connecting $(0,1)$ and $(1,0)$ to (α,α). In fact, B_δ spreads the following masses uniformly on the indicated line segments: β on $\triangle_{[0,\alpha]}$; $\beta - 2\alpha + 1$ on $\triangle_{[\alpha,1]}$; and $\alpha - \beta$ on both of the line segments connecting $(0,1)$ and $(1,0)$ to (α,α).

2. Supports of Bertino copulas

Fix a diagonal δ and let B denote its associated Bertino copula. Define $h : I \to I$ by

$$h(u) = \max\{s \geq u : \hat{\delta}(t) \geq \hat{\delta}(u) \text{ for all } t \in [u, s]\}. \qquad (2.1)$$

Note that $[u, h(u)]$ is the largest interval with left-hand endpoint u on which $\hat{\delta} \geq \hat{\delta}(u)$. Some obvious properties of h follow.

Proposition 2.1. *$h(u) \geq u$ for all $u \in I$; $\hat{\delta} \circ h = \hat{\delta}$ on I; h is right-continuous on I; h is strictly decreasing on intervals on which $\hat{\delta}$ is strictly increasing; h is constant on intervals on which $\hat{\delta}$ is constant; and h is the identity on open intervals on which $\hat{\delta}$ is strictly decreasing; and the converses of the preceding three statements hold.*

Consider an open interval (a, b) on which h is strictly decreasing and continuous. Let $(c, d) = h((a, b))$ and note that $\hat{\delta}(a) = \hat{\delta}(d)$ and $\hat{\delta}(b) = \hat{\delta}(c)$. Let h^{-1} denote the inverse of the restriction $h|_{(a,b)}$ of h to (a, b), set

$$\begin{aligned} S &= \{(u, v) \in (a, b) \times I : \quad u \leq v \leq h(u)\}, \\ T &= \{(u, v) \in I \times (c, d) : \quad h^{-1}(v) \leq u \leq v\}, \end{aligned}$$

and let S' and T' be the reflections of S and T, respectively, about $\triangle$. Note from the definition of h that

$$B(u, v) = \begin{cases} u - \hat{\delta}(u) & \text{if } (u, v) \in S, \\ u - \hat{\delta}(v) & \text{if } (u, v) \in T, \\ v - \hat{\delta}(v) & \text{if } (u, v) \in S', \\ v - \hat{\delta}(u) & \text{if } (u, v) \in T'. \end{cases}$$

Thus the μ_B–measure of any rectangle in S, T, S', or T' is zero. Now, if $[\alpha, \beta]$ is a subinterval of (a, b), then

$$\begin{aligned} \mu_B(\text{graph } \ h|_{[\alpha,\beta]}) &= \mu_B([\alpha, \beta] \times [h(\beta), h(\alpha)]) \\ &= \beta - \hat{\delta}(h(\alpha)) + \alpha - \hat{\delta}(\alpha) - (\beta - \hat{\delta}(\beta)) - (\alpha - \hat{\delta}(\alpha)) \\ &= \hat{\delta}(\beta) - \hat{\delta}(\alpha), \end{aligned}$$

as $(\alpha, h(\beta)) \in S$ and $(\beta, h(\alpha)) \in T$. It follows from Proposition 2.2 that the closure of the graph of $h|_{(a,b)}$ lies in the support of B. For any $[\alpha, \beta] \subseteq (a, b)$, we also have

$$\begin{aligned} \mu_B(\triangle_{[\alpha,\beta]}) &= \mu_B([\alpha, \beta]^2) \\ &= \beta - \hat{\delta}(\beta) + \alpha - \hat{\delta}(\alpha) - (\alpha - \hat{\delta}(\alpha)) - (\alpha - \hat{\delta}(\alpha)) \\ &= \beta - \alpha - (\hat{\delta}(\beta) - \hat{\delta}(\alpha)) = \delta(\beta) - \delta(\alpha). \end{aligned}$$

Thus, $\triangle_{(\alpha,\beta)}$ does not intersect the support of B if and only if $\hat{\delta}$ has slope 1 on $[\alpha,\beta]$. Note that the μ_B–measure of the union of $\triangle_{(a,b)}$ and the graph of $h|_{(a,b)}$ is $\delta(b)-\delta(a)+\hat{\delta}(b)-\hat{\delta}(a)=b-a$, which is the μ_B–measure of $(a,b)\times I$. If $[\alpha,\beta]$ is a subinterval of (c,d), then

$$\begin{aligned}\mu_B(\triangle_{[\alpha,\beta]}) &= \beta-\hat{\delta}(\beta)+\alpha-\hat{\delta}(\alpha)-(\alpha-\hat{\delta}(\beta))-(\alpha-\hat{\delta}(\beta))\\ &= \beta-\alpha-(\hat{\delta}(\alpha)-\hat{\delta}(\beta)).\end{aligned}$$

In this case, $\triangle_{(\alpha,\beta)}$ does not intersect the support of B if and only if $\hat{\delta}$ has slope -1 on $[\alpha,\beta]$. Note that the μ_B–measure of the union of $\triangle_{(c,d)}$ and the graph of $h|_{(a,b)}$ is $d-c-(\hat{\delta}(c)-\hat{\delta}(d))+\hat{\delta}(b)-\hat{\delta}(a)=d-c$, which is the μ_B–measure of $I\times(c,d)$.

Since $\hat{\delta}$ is constant on any interval in the complement of the closure of the union of all pairs of intervals (a,b) and (c,d) of the form considered in the preceding paragraph, we now suppose that $\hat{\delta}$ has constant value r on the interval $[p,q]$. Then $B(u,v)=\min(u,v)-r$ for each $(u,v)\in[p,q]^2$, so rectangles in $[p,q]^2$ which do not meet $\triangle$ have μ_B–measure zero, and if $[\alpha,\beta]$ is a subinterval of $[p,q]$, then

$$\mu_B(\triangle_{[\alpha,\beta]})=\mu_B([\alpha,\beta]^2)=\beta-r+\alpha-r-(\alpha-r)-(\alpha-r)=\beta-\alpha.$$

Thus, $\triangle_{[p,q]}$ lies in the support of B and $\mu_B(\triangle_{[p,q]})=q-p$, which is the μ_B–measure of $[p,q]\times I$.

The preceding paragraphs and the symmetry of the support of a symmetric copula establish the following.

Theorem 2.2. *The support of the Bertino copula B_δ is the smallest closed set which is symmetric with respect to $\triangle$ and contains the continuous, strictly decreasing parts of the graph of h and the part of $\triangle$ corresponding to the complement of the union of the intervals on which $\hat{\delta}$ has slope ± 1.*

Corollary 2.3. *Bertino copulas are singular.*

Example 2.4. Let δ be the diagonal which is piecewise linear with slope 0 on $[0,\frac{1}{4}]$, slope 1 on $[\frac{1}{4},\frac{3}{4}]$ and slope 2 on $[\frac{3}{4},1]$. Then $\hat{\delta}$ is piecewise linear with slope 1 on $[0,\frac{1}{4}]$, slope 0 on $[\frac{1}{4},\frac{3}{4}]$ and slope -1 on $[\frac{3}{4},1]$. Clearly $h(t)=1-t$ on $[0,\frac{1}{4}]$, so the support of B_δ consists of the three line segments between $(0,1)$ and $(\frac{1}{4},\frac{3}{4})$, between $(\frac{1}{4},\frac{1}{4})$ and $(\frac{3}{4},\frac{3}{4})$, and between $(\frac{3}{4},\frac{1}{4})$ and $(1,0)$. This support set uniquely determines B_δ.

Definition 2.5. A *Bertino set* is a closed subset S of I^2 which is symmetric with respect to $\triangle$, consists of the union of $\triangle$ and a collection of graphs of continuous, strictly decreasing functions, and satisfies the property:

$$\text{if } (u,v)\in S \text{ and } u<v, \text{ then } S\cap(u,v)\times(v,1) \text{ is empty.} \qquad (2.2)$$

Since Bertino sets are symmetric with respect to $\triangle$, they also satisfy the property

$$\text{if } (u,v) \in S \text{ and } u > v, \text{ then } (u,1) \times (v,u) \cap S \text{ is empty.} \tag{2.3}$$

If the graph of a continuous, strictly decreasing function g lies in a Bertino set S as pictured in Figure 1, then (2.2) and (2.3) imply that no point of S lies on the four unshaded regions of the same figure.

In particular note that there exists a (possibly infinite) partition P of I such that, for each open interval $J \in P$, $S \cap J \times I$ and $S \cap I \times J$ are both either $\triangle_J$, or the union of $\triangle_J$ and the graph of a continuous strictly decreasing function which disconnects the strip.

g

Figure 1

Lemma 2.6. *If C is a copula with support in a Bertino set S, then C is symmetric.*

Proof: Let J be any open interval for which $S \cap J \times I = \triangle_J \cup$ graph(g), where g is a continuous, strictly decreasing function without fixed points on J. Since $S \cap I \times J = \triangle_J \cup$ graph(g^{-1}), we see that the way in which C spreads mass on $\triangle_J$ uniquely determines the way in which C spreads mass on the graphs of g and g^{-1} and that it is done symmetrically. (Note that this, in turn, uniquely determines the way in which C spreads mass on $\triangle_{g(J)}$.) Since the rest of the mass is spread on $\triangle$, it follows that C is symmetric.

Lemma 2.7. *Two copulas are identical if they have the same diagonal section and support in the same Bertino set S.*

Proof: Note from the proof of the preceding lemma that a copula with support in S is uniquely determined by the way in which mass is spread on the part of $\triangle$ corresponding to the union of the open intervals J for which $S \cap J \times I = \triangle_J \cup$ graph(g), with g a continuous, strictly decreasing function satisfying $g(t) > t$ for all $t \in J$. It is obvious from Figure 1 that the way in which mass is spread on $\triangle_J$ for such an interval J is uniquely determined by the values of the diagonal section on J.

Theorem 2.8. *A copula is a Bertino copula if and only if its support lies in a Bertino set.*

Proof: Let δ be a diagonal and let S denote the union of $\triangle$ and the support of B_δ. To show that S is a Bertino set, it suffices by Theorem 2.2 to establish (2.2). Fix u in an open interval on which h is continuous and strictly decreasing. Let $v = h(u)$. For each $s \in (u,v)$, $\hat{\delta}(s) > \hat{\delta}(u) = \hat{\delta}(v)$, so $h(s) < v$. Thus, there is no point of the graph of h in $(u,v) \times [v,1]$. We obtain the desired result when taking the closure of these sets.

For the converse, suppose that C is a copula with support in a Bertino set S. Let $\delta = \delta_C$ and let h be defined as usual from $\hat{\delta}$. Let $J = (a, b)$ be an interval for which $S \cap J \times I = \triangle_J \cup$ graph(g), with g as in Figure 1. Let $(c, d) = g(J)$. Note that $\hat{\delta}(t) = \mu_C([0, t] \times [t, 1])$ is nondecreasing on $[a, b]$, $\hat{\delta} \geq \hat{\delta}(b)$ on $[b, c]$, $\hat{\delta}(c) = \hat{\delta}(b)$, and $\hat{\delta}$ is nonincreasing on $[c, d]$. If J' is an open subinterval of J for which the graph of $g|_{J'}$ is contained in the support of C, then $\hat{\delta}$ is strictly increasing on J' and $h|_{J'} = g|_{J'}$. On the other hand, if the graph of $g|_{J'}$ does not meet the support of C, then $\hat{\delta}$ and h are constant on J'. Finally, if J is an open interval for which $S \cap J \times I = \triangle_J$, then $\hat{\delta}$ and h are again constant on J. It follows from symmetry and Theorem 2.2 that the support of $B = B_\delta$ lies in S. Since $\delta_B = \delta_C$, the result follows from the preceding lemma.

Example 2.9. An X-*copula* is a copula with support in the union of the two diagonals of the unit square. It follows from the preceding theorem that all X-copulas are Bertino copulas. Moreover, B_δ is an X-copula if and only if $\hat{\delta}$ is nondecreasing on $[0, 1/2]$ and symmetric with respect to $1/2$, as it is only in these cases that $h(t) = 1 - t$ on all open intervals on which $\hat{\delta}$ is strictly increasing. Consideration of various diagonals establishes that mass can be spread arbitrarily (subject to the limits of uniform margins) on any of the four line segments connecting one of the corners of the unit square to the point $(1/2, 1/2)$ and that this then uniquely determines an X-copula.

3. Statistical characterizations

Theorem 3.1. *Let U and V be random variables uniformly distributed on I with copula C. Then C is a Bertino copula if and only if for each $u, v \in I$, there exists $t \in [\{u, v\}]$ such that*

$$\begin{aligned} &P[\min(U, V) \leq \min(u, v) \text{ and } \max(U, V) > \max(u, v)] \\ &\quad = P[\min(U, V) \leq t < \max(U, V)]. \end{aligned} \tag{3.1}$$

Proof: Let $C = B_\delta$. Fix $u \leq v$ in I and t in $[u, v]$ for which $\hat{\delta}(t)$ is the minimum value of $\hat{\delta}$ on $[u, v]$. Then

$$\begin{aligned} &P[\min(U, V) \leq \min(u, v) \text{ and } \max(U, V) > \max(u, v)] \\ &\quad = P[U \leq u,\ V > v] + P[V \leq u,\ U > v] \\ &\quad = u - B_\delta(u, v) + u - B_\delta(v, u) \\ &\quad = \hat{\delta}(t) + \hat{\delta}(t) \\ &\quad = P[U \leq t < V] + P[V \leq t < U] \\ &\quad = P[\min(U, V) \leq t < \max(U, V), \end{aligned}$$

so (3.1) holds. The proof for $v \le u$ in I is obvious.

Fix $u \le v$ in I and assume there exists $t \in [u, v]$ such that (3.1) holds. Then $u - C(u,v) + u - C(v,u) = \hat{\delta}_C(t) + \hat{\delta}_C(t)$, so $\mu_C([0,u] \times [v,1] \cup [v,1] \times [0,u]) = \mu_C([0,t] \times [t,1] \cup [t,1] \times [0,t])$ and hence $\mu_C([0,t] \times [t,1] \setminus [0,u] \times [v,1]) = 0$ and $\mu_C([t,1] \times [0,t] \setminus [v,1] \times [0,u]) = 0$. Clearly, $C(u,v) = u - \hat{\delta}_C(t)$, $C(v,u) = u - \hat{\delta}_C(t)$ and $\hat{\delta} \ge \hat{\delta}(t)$ on $[u,v]$. Hence, for any $(u,v) \in I^2$, $C(u,v) = \min(u,v) - \hat{\delta}_C(t) = B_\delta(u,v)$, where $\delta = \delta_C$.

A Bertino copula B_δ is *simple* if for each (u,v) in I^2

$$B_\delta(u,v) = \min(u,v) - \min(\hat{\delta}(u),\ \hat{\delta}(v)).$$

Note that this is the case if and only if $\hat{\delta}$ has the following *nondecreasing/nonincreasing property*: $\hat{\delta}$ is nondecreasing on $[0,\alpha]$ and nonincreasing on $[\alpha,1]$ for some $\alpha \in (0,1)$. Thus, X-copulas are simple Bertino copulas.

When $t = \min(u,v)$ in the preceding theorem, we obtain $C(u,v) = \min(u,v) - \hat{\delta}_C(\min(u,v)) = \delta_C(\min(u,v))$, which is equivalent to $P[U \le u, V \le v] = P[\max(U,V) \le \min(u,v)]$. When $t = \max(u,v)$ in the preceding theorem, we obtain $C(u,v) = \min(u,v) - \hat{\delta}_C(\max(u,v))$, which is equivalent to $P[U > u, V > v] = P[\max(U,V) > \max(u,v)]$. This establishes

Corollary 3.2. *Let U and V be random variables uniformly distributed on I with copula C. Then C is a simple Bertino copula if and only if, for each $(u,v) \in I^2$, either $P[U \le u, V \le v] = P[\max(U,V) \le \min(u,v)]$ or $P[U > u, V > v] = P[\min(U,V) > \max(u,v)]$.*

Proposition 3.3. *X-copulas are the only copulas that can be written in the form $M - f(M - W)$ with f nondecreasing on $[0,1/2]$.*

Proof: If $C = B_\delta$ is an X-copula, then $\hat{\delta}$ is nondecreasing on $[0,1/2]$ and $C = M - \hat{\delta}(M - W)$ on I^2. For the converse, suppose that f is nondecreasing on $[0,1/2]$ and that $C = M - f(M - W)$ is a copula. Then $\hat{\delta}_C(t) = f(t)$ if $t \in [0,1/2]$ and $\hat{\delta}_C(t) = f(1-t)$ if $t \in [1/2,1]$, so $B = B_{\delta_C}$ is an X-copula. If $u \le v$, then

$$\min_{t \in [u,v]} \hat{\delta}_C(t) = \left\{ \begin{matrix} f(u) & \text{if } u + v \le 1 \\ f(1-v) & \text{if } u + v \ge 1 \end{matrix} \right\} = f(M - W)(u,v).$$

Clearly, $B = M - f(M - W)$ by symmetry.

4. Extremality

A copula C is *extremal* if it cannot be written as a nontrivial convex sum, i.e., if $C = \alpha C_1 + \beta C_2$ with C_i copulas, $\alpha,\ \beta > 0$ and $\alpha + \beta = 1$, then $C_1 = C_2 = C$. Note that if $C = \alpha C_1 + \beta C_2$ as above, then the supports of C_1 and C_2 are contained in the support of C. Thus, if C is a Bertino copula, then so are C_1 and C_2.

Theorem 4.1. *The following are equivalent for a Bertino copula $B = B_\delta$:*

(a) *B is extremal;*

(b) *B is uniquely determined by its support set; and*

(c) *Each interval on which $\hat{\delta}$ is strictly increasing can be partitioned into a possibly infinite number of open intervals such that for each interval J in that partition, either $\hat{\delta}$ has slope 1 on J or $\hat{\delta}$ has slope -1 on $h(J)$.*

Proof: Assume (c). It follows from Theorem 2.3 that for each such J, either $\mu_B(\triangle_J) = 0$ or $\mu_B(\triangle_{h(J)}) = 0$. Hence there is only one way to spread mass on graph $h|_J$ and hence only one way to spread mass on

$$J \times I \cup I \times J \cup h(J) \times I \cup I \times h(J).$$

Clearly (b) holds. That (b) implies (a) is obvious.

Before finishing the proof of Theorem 4.1, we will examine a process for shifting mass on the support of a Bertino copula $B = B_\delta$. Let $J = (a, b)$ be an interval on which $\hat{\delta}$ is strictly increasing and $h(J) = (c, d)$, and let γ be a positive number. For each subinterval K of J, we require that graph $h|_K$ and graph $(h|_K)^{-1}$ have mass $\gamma\mu_B(\text{graph } h|_K)$; that $\triangle_K$ have mass $\mu_B(K) + (1-\gamma)\mu_B(\text{graph } (h|_K)$; and that $\triangle_{h(K)}$ have mass $\mu_B(h(K)) + (1-\gamma)\mu_B(\text{graph } h|_K)$. Note that the masses of $K \times I$, $I \times K$, $h(K) \times I$ and $I \times h(K)$ remain the same as before. When $\gamma < 1$, mass is moved from graph $h|_J$ to $\triangle_J$ and $\triangle_{h(J)}$, and the result is another Bertino copula by the preceding sentence and Theorem 2.8. When $\gamma > 1$, mass is moved from $\triangle_J$ and $\triangle_{h(J)}$ to graph $h|_J$, so there must be enough mass on $\triangle$ initially. Specifically, for each subinterval $K = (\alpha, \beta)$ of J, we need

$$\begin{aligned}(\gamma - 1)(\hat{\delta}(\beta) - \hat{\delta}(\alpha)) &= (\gamma - 1)\mu_B(\text{graph } h|_K) \le \mu_B(\triangle_K) \\ &= \delta(\beta) - \delta(\alpha),\end{aligned}$$

and for each $\alpha = h(t) < \beta = h(s) \in h(J)$, we need

$$\begin{aligned}(\gamma - 1)(\hat{\delta}(\alpha) - \hat{\delta}(\beta)) &= (\gamma - 1)\mu_B(\text{graph } h|_{(s,t)}) \le \mu_B(\triangle_{(\alpha,\beta)}) \\ &= \beta - \alpha - (\hat{\delta}(\alpha) - \hat{\delta}(\beta)).\end{aligned}$$

Thus we obtain a Bertino copula in the case $\gamma > 1$ if and only if

$$\begin{array}{c}\gamma(\hat{\delta}(\beta) - \hat{\delta}(\alpha)) \leq \beta - \alpha \text{ whenever } \alpha < \beta \in J \textit{ and} \\ \gamma(\hat{\delta}(\alpha) - \hat{\delta}(\beta)) \leq \beta - \alpha \text{ whenever } \alpha < \beta \in h(J).\end{array} \tag{4.1}$$

Let $B_\gamma(J)$, or simply B_γ, denote the resulting Bertino copula in either case. Letting δ_γ denote the diagonal section of B_γ and recalling that $\hat{\delta}_\gamma(t) = \mu_{B_\gamma}([0,t] \times [t,1])$ and $\hat{\delta}_\gamma \circ h = \hat{\delta}_\gamma$, it is easy to see that

$$\hat{\delta}_\gamma(t) = \begin{cases} \hat{\delta}(t) & \text{if } t \in [0,a] \cup [d,1] \\ \gamma\hat{\delta}(t) + (1-\gamma)\hat{\delta}(a) & \text{if } t \in [a,b] \cup [c,d] \\ \hat{\delta}(t) - (1-\gamma)(\hat{\delta}(b) - \hat{\delta}(a)) & \text{if } t \in [b,c]. \end{cases} \tag{4.2}$$

Returning to the proof of Theorem 4.1, assume that (c) does not hold. It follows from Proposition 1.1 that there exists an interval $J = (a,b)$, on which $\hat{\delta}$ is strictly increasing, for which $h(J)$ is an interval (c,d) and the secant-slope function $m(\alpha,\beta) = \frac{\hat{\delta}(\beta)-\hat{\delta}(\alpha)}{\beta-\alpha}$ satisfies $m < 1$ for pairs of points on J and $m > -1$ for pairs of points on $h(J)$. Note that m is bounded away from 1 for pairs of points on some subinterval of J. [Otherwise, the derivative of $\hat{\delta}$ is 1 at all points of J where $\hat{\delta}$ is differentiable. Since $\hat{\delta}$ is strictly increasing on J, it is differentiable almost everywhere on J. Hence $\hat{\delta}$ has slope 1 on J — a contradiction.] Assume that the subinterval is J. Using a similar argument on $h(J)$, and renaming J, if necessary, we see that m is bounded away from 1 for all pairs of points on J and bounded away from -1 for all pairs of points on $h(J)$ — specifically, there exists a $\gamma \in (1,2)$ such that

$$\begin{aligned}\gamma m(\alpha,\beta) &\leq 1 \text{ whenever } \alpha,\beta \in J \textit{ and} \\ \gamma m(\alpha,\beta) &\geq -1 \text{ whenever } \alpha,\beta \in h(J),\end{aligned}$$

i.e., such that (4.1) holds. Let $C_1 = B_\gamma$, $C_2 = B_{2-\gamma}$, and note from (4.2) that $\frac{1}{2}\hat{\delta}_\gamma + \frac{1}{2}\hat{\delta}_{\gamma-2} = \hat{\delta}$. Hence $B = \frac{1}{2}C_1 + \frac{1}{2}C_2$ as desired.

References

Bertino, S.: 1977, 'Sulla dissomiglianza tra mutabili cicliche'. *Metron* **35**, 53–88.

Fredricks, G. A. and R. B. Nelsen: 1997, *'Copulas constructed from diagonal sections'. In: V. Beneŝ and J. Ŝtêpán (eds.)*: Distributions with Given Marginals and Moment Problems, pp. 129–136. *Dordrecht: Kluwer Academic Publishers.*

Nelsen, R. B.: 1999, *An Introduction to Copulas. New York: Springer-Verlag.*