Multivariate Archimedean Quasi-copulas

Roger B. Nelsen
Department of Mathematical Sciences, Lewis & Clark College, Portland, Oregon, USA.

José Juan Quesada Molina
Departamento de Matemática Aplicada, Universidad de Granada, Granada, Spain.

José Antonio Rodríguez Lallena
Manuel Úbeda Flores
Departamento de Estadística y Matemática Aplicada, Universidad de Almería, Almería, Spain.

ABSTRACT

Every multivariate distribution function with continuous marginals can be represented in terms of an unique n-copula, that is, in terms of a distribution function on $[0,1]^n$ with uniform marginals. The notion of quasi-copula was introduced in [1] by Alsina *et al.* (1993) and was used by these authors and others to characterize operations on distribution functions that can or cannot be derived from operations on random variables. In [4] and [5], Genest *et al.* (1999) and (2000) have characterized the concept of quasi-copula and multivariate quasi-copula in simpler operational terms. This paper is concerned with a special class of multivariate quasi-copulas called Archimedean, which includes the uniform representation of many multivariate distributions, and other interesting functions such as $W^n(u_1, u_2, ..., u_n) = \max\{0, u_1 + u_2 + ... + u_n - n + 1\}$, the best-possible lower bound for n-copulas. We present the basic properties of Archimedean n-quasi-copulas. If they are n-copulas, we investigate their support and provide a relationship between the generator of an Archimedean n-copula and a measure of multivariate association based on the notion of average multivariate total positivity of order two.

AMS 1991 *Subject Classifications*: Primary 60E05; Secondary 62H05, 62E10. (?)

Keywords: Archimedean quasi-copulas, Quasi-copulas, Lipschitz condition, Copulas, Measures of multivariate association.

1 Introduction.

The term "copula", coined in [12] by Sklar (1959), is now common in the statistical literature. The importance of copulas as a tool for statistical analysis and modelling stems largely from the observation that the joint distribution H of a set of $n \geq 2$ random variables X_i with marginals F_i can be expressed in the form

$$H(x_1, x_2, ..., x_n) = C\{F_1(x_1), F_2(x_2), ..., F_n(x_n)\}$$

in terms of a copula C that is uniquely determined on the set $\text{Ran}F_1 \times ... \times \text{Ran}F_n$. For more details see [8] and [11].

In [1], Alsina *et al.* (1993) have introduced recently the notion of "quasi-copula" in order to show that a certain class of operations on univariate distribution functions is not derivable from corresponding operations on random variables defined on the same probability space. The same concept was also used in [10] by Nelsen *et al.* (1996) to characterize, in a given class of operations on distribution functions, those that do derive from corresponding operations on random variables.

In [4], Genest *et al.* (1999) have characterized bivariate quasi-copulas, and, in [5], the same authors characterize multivariate quasi-copulas. The main contribution of the last paper is the following: A function $Q : \mathbf{I}^n \longrightarrow \mathbf{I}$ ($\mathbf{I}$=[0,1]) is an *n-quasi-copula* if, and only if,

(a) $Q(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) = 0$ and $Q(1, ..., 1, x_i, 1, ..., 1) = x_i$ for all $(x_1, x_2, ..., x_n)$ in $\mathbf{I}^n$;

(b) Q is non-decreasing in each variable;

(c) Q satisfies the Lipschitz condition

$$|Q(x_1, x_2, ..., x_n) - Q(x'_1, x'_2, ..., x'_n)| \leq \sum_{i=1}^{n} |x_i - x'_i|$$

for all $(x_1, x_2, ..., x_n)$ and $(x'_1, x'_2, ..., x'_n)$ in $\mathbf{I}^n$.

Of course, every n-copula is an n-quasi-copula. When Q is an n-quasi-copula but it is not an n-copula, we say that Q is a *proper* n-quasi-copula.

In this paper we discuss an important class of multivariate quasi-copulas called Archimedean n-quasi-copulas. These n-quasi-copulas could find a wide range of applications for a number of reasons: (1) The ease with which they can be constructed; (2) The great variety of families of n-quasi-copulas which belong to this class; and (3) The many nice properties possessed by the members of this class. One of the most important Archimedean n-quasi-copulas is the fundamental best-possible lower bound for n-copulas (see [8]). For a wide study of bivariate Archimedean copulas, see [2], [3] and [8].

Of course, the properties about n-quasi-copulas that we show in this contribution will have interest when we work with n-copulas. But they are also interesting properties for proper n-quasi-copulas whose study is developing in the last years (see [1], [4], [5], [9] and [10]).

In Section 2 we define and characterize Archimedean n-quasi-copulas and give some nice properties of these functions; moreover, we show that an ordering of two Archimedean n-quasi-copulas is determined by properties of the generators. In Section 3 we investigate the support of an Archimedean n-copula, illustrate the existence of distributions with singular components and find a relationship between a measure of multivariate association based on the notion of average multivariate total positivity of order two and the generator of an Archimedean n-copula. We conclude this communication with an open problem in Section 4.

2 Multivariate Archimedean quasi-copulas: definition and main properties.

Let ϕ be a continuous, strictly decreasing function from $\mathbf{I}$ to $[0,\infty]$ such that $\phi(1)=0$. The *pseudo-inverse* of ϕ is the function $\phi^{[-1]}$ with Dom$\phi^{[-1]}=[0,\infty]$ and Ran$\phi^{[-1]}=\mathbf{I}$ given by

$$\phi^{[-1]}(t)=\begin{cases}\phi^{-1}(t), & 0\le t\le \phi(0),\\ 0, & \phi(0)\le t\le\infty.\end{cases} \tag{1}$$

LEMMA 2.1. *Let ϕ be a continuous, strictly decreasing function from $\mathbf{I}$ to $[0,\infty]$ such that $\phi(1)=0$, and let $\phi^{[-1]}$ be the pseudo-inverse of ϕ defined by* (1). *Let Q be the function from $\mathbf{I}^n$ to $\mathbf{I}$ given by*

$$Q(u_1,u_2,...,u_n)=\phi^{[-1]}(\phi(u_1)+\phi(u_2)+...+\phi(u_n)). \tag{2}$$

Then Q satisfies both the conditions (a) *and* (b) *for the definition of n-quasi-copula.*

DEFINITION 2.2. Let ϕ be a function satisfying the hypotheses of Lemma 2.1. Let Q be the function defined by (2). If Q is an n-copula (respectively, n-quasi-copula), then we say that Q is an *Archimedean n-copula* (respectively, *n-quasi-copula*). The function ϕ is called a *generator* of the n-copula (respectively, n-quasi-copula). If $\phi(0)=\infty$, we say that ϕ is a *strict* generator. In this case, $\phi^{[-1]}=\phi^{-1}$ and Q is said to be a *strict* Archimedean n-copula (respectively, n-quasi-copula).

The main result of this section is the following theorem:

THEOREM 2.3. *Let ϕ be a continuous, strictly decreasing function from $\mathbf{I}$ to $[0,\infty]$ such that $\phi(1)=0$, and let $\phi^{[-1]}$ be the pseudo-inverse of ϕ defined by* (1). *Then, the function Q from $\mathbf{I}^n$ to $\mathbf{I}$ given by* (2) *is an n-quasi-copula if, and only if, ϕ is convex.*

If $n \geq 3$ there exist Archimedean n-quasi-copulas that are not n-copulas. One example is the proper Archimedean n-quasi-copula $W^n(u_1, u_2, ..., u_n) = \max(0, u_1 + u_2 + ... + u_n - n + 1)$, since its generator $\phi(t) = 1 - t$ is convex. Note that for the bivariate case, there do not exist proper Archimedean quasi-copulas since the 2-increasingness is equivalent to the Lipschitz condition (see [8]). This equivalence is false for the n-increasingness with $n > 2$.

For the next theorem, we need first two definitions.

For an Archimedean n-quasi-copula Q and for $t > 0$, it is clear that the level set $\{(u_1, u_2, ..., u_n) \in \mathbf{I}^n | Q(u_1, u_2, ..., u_n) = t\}$ consists of the points $(u_1, u_2, ..., u_n)$ in $\mathbf{I}^n$ such that $\phi(u_1) + \phi(u_2) + ... + \phi(u_n) = \phi(t)$.

Next, we define the diagonal section of an n-quasi-copula Q.

DEFINITION 2.4. Let Q be an n-quasi-copula. The *diagonal section* of Q is the function δ_Q from $\mathbf{I}$ to $\mathbf{I}$ defined by $\delta_Q(u) = Q(u, u, ..., u)$.

Now, we are in position to state the following theorem, which summarizes the first properties of Archimedean n-quasi-copulas (they are simple generalizations of those of Archimedean 2-copulas).

THEOREM 2.5. *Let Q be an Archimedean n-quasi-copula with generator ϕ. Then:*

(i) *For u in $(0, 1)$, $\delta_Q(u) < u$.*

(ii) *If $c > 0$ is any constant, then $c\phi$ is also a generator of Q.*

(iii) *If π denotes any permutation of $\{1, 2, ..., n\}$, then $Q(u_{\pi(1)}, u_{\pi(2)}, ..., u_{\pi(n)}) = Q(u_1, u_2, ..., u_n)$.*

(iv)*If π and π' denote any permutations of $\{1, 2, ..., 2n - 1\}$, then Q is associative in the following sense:*

$$Q(u_{\pi(1)}, ..., u_{\pi(i-1)}, Q(u_{\pi(i)}, ..., u_{\pi(i+n-1)}), u_{\pi(i+n)}, ..., u_{\pi(2n-1)}) =$$

$$Q(u_{\pi'(1)}, ..., u_{\pi'(j-1)}, Q(u_{\pi'(j)}, ..., u_{\pi'(j+n-1)}), u_{\pi'(j+n)}, ..., u_{\pi'(2n-1)})$$

for all $i, j \in \{1, 2, ..., n\}$.

(v) *Q is strict if, and only if, $Q(u_1, u_2, ..., u_n) > 0$ for all $(u_1, u_2, ..., u_n) \in (0, 1]^n$.*

(vi) *For almost $u_1, u_2, ..., u_n$ in $\mathbf{I}$,*

$$\frac{\phi'(u_1)}{\frac{\partial}{\partial u_1} Q(u_1, u_2, ..., u_n)} = \frac{\phi'(u_2)}{\frac{\partial}{\partial u_2} Q(u_1, u_2, ..., u_n)} = ... = \frac{\phi'(u_n)}{\frac{\partial}{\partial u_n} Q(u_1, u_2, ..., u_n)}.$$

(vii) *The level sets of Q are convex.*

Now, we show that under certain conditions of the generators of two Archimedean n-quasi-copulas, they are ordered. But first, we need some definitions and notation.

Let Q_1 and Q_2 be two Archimedean n-quasi-copulas. $Q_1 \le Q_2$ denotes that $Q_1(\mathbf{u}) \le Q_2(\mathbf{u})$ for all $\mathbf{u}$ in $\mathbf{I}^n$. This defines an ordering in the set of n-quasi-copulas similar to an orthant ordering for multivariate copulas (see Section 3 or [8] for details).

It is well-known that a function f defined on $[0,\infty)$ is *subadditive* if, for all x, y in $[0,\infty)$, $f(x+y) \le f(x) + f(y)$. We need the following simple lemma:

LEMMA 2.6. *Let f be a function defined on $[0,\infty)$ such that $f(0) = 0$. Then f is subadditive if, and only if, $f(x_1 + x_2 + ... + x_n) \le f(x_1) + f(x_2) + ... + f(x_n)$ for all $x_1, x_2, ..., x_n$ in $[0,\infty)$.*

We state now a theorem that characterizes the ordering of Archimedean n-quasi-copulas in terms of composites of generators and their inverses.

THEOREM 2.7. *Let Q_1 and Q_2 be Archimedean n-quasi-copulas generated, respectively, by ϕ_1 and ϕ_2. Then $Q_1 \le Q_2$ if, and only if, $\phi_1 \circ \phi_2^{[-1]}$ is subadditive.*

By using several well-known sufficient conditions for the subadditivity (see [2] and [8]), we have the following corollary:

COROLLARY 2.8. *Under the hypotheses of Theorem* 2.7:

(A) *If $\phi_1 \circ \phi_2^{[-1]}$ is concave, then $Q_1 \le Q_2$.*

(B) *If ϕ_1/ϕ_2 is non-decreasing on $(0,1)$, then $Q_1 \le Q_2$.*

(C) *If ϕ_1 and ϕ_2 are continuously differentiable on $(0,1)$, and if ϕ_1'/ϕ_2' is non-decreasing on $(0,1)$, then $Q_1 \le Q_2$.*

As a consequence of this corollary, if Q is an Archimedean n-quasi-copula with generator ϕ, then $Q \ge \prod$ whenever one of the following conditions is satisfied: 1) $-ln\phi^{-1}(t)$ is concave on $(0,\infty)$; 2) $-lnt/\phi(t)$ is non-decreasing on (0,1); and 3) ϕ is continuously differentiable on (0,1) and $-1/(t\phi'(t))$ is non-decreasing on (0,1).

As we noted in the introduction, there are certain properties that have interest in the set of Archimedean n-copulas. We refer them in the next section.

3 Other properties of Archimedean n-copulas.

First, it is interesting to recall (see [8]) that a sufficient condition for Q be an n-copula for $2 \le n \le m$ is that $(-1)^k \frac{d^k}{dt^k}\phi^{[-1]}(t) \ge 0$ for all $t \in [0,\infty)$ and $k = 1, 2, ..., m$. In this case, it is said that $\phi^{[-1]}$ is *m-monotone*.

For any n-copula C, let $C(u_1, u_2, ..., u_n) = A_C(u_1, u_2, ..., u_n) + S_C(u_1, u_2, ..., u_n)$ where $A_C(u_1, u_2, ..., u_n) = \int_0^{u_1}\int_0^{u_2} ... \int_0^{u_n} \frac{\partial^n}{\partial v_1 \partial v_2 ... \partial v_n} C(v_1, v_2, ..., v_n) dv_1 dv_2 ... dv_n$. If $C \equiv A_C$ then C is *absolutely continuous*, and if $C \equiv S_C$ then C is *singular*. The C-measure of the absolutely continuous component is $A_C(1, 1, ..., 1)$, and the C-measure of the singular component is $S_C(1, 1, ..., 1)$.

The C-measure carried by each of the level sets of an Archimedean n-copula C under certain hypotheses on the generator ϕ is given in the following theorem:

THEOREM 3.1. *Let C_n be the Archimedean n-copula generated by a function ϕ such that $\phi^{[-1]}$ is m-monotone. Then the C_n-measure of the set $\{(u_1, u_2, ..., u_n) \in \mathbf{I}^n | C_n(u_1, u_2, ..., u_n) \leq t\}$, which we denote by $K_n(t)$, is given by*

$$K_n(t) = t + \frac{1}{\phi'(t)} \sum_{i=1}^{n-1} \frac{(-1)^i}{i!} [\phi(t)]^i \alpha_{i-1}(t), \tag{3}$$

where $\alpha_i(t) = (\frac{\alpha_{i-1}(t)}{\phi'(t)})'$ for all $i = 1, 2, ..., n-2$, with $\alpha_0(t) = 1$.

The following corollary presents a probabilistic interpretation of Theorem 3.1 which will be useful when in Corollary 3.3 we consider a nonparametric measure of multivariate association for Archimedean n-copulas.

COROLLARY 3.2. *Let $U_1, U_2, ..., U_n$ be uniform $(0, 1)$ random variables whose joint distribution function is the Archimedean n-copula C generated by a function ϕ such that $\phi^{[-1]}$ is m-monotone. Then the function K_n given by (3) is the distribution function of the random variable $C(U_1, U_2, ..., U_n)$.*

As an application of Theorem 3.1 and Corollary 3.2, we provide an expression for the nonparametric measure of multivariate association $\tau_{n,C}$ derived from average multivariate total positivity of order two of C (see [7] for details). This measure is given by

$$\tau_{n,C} = \frac{1}{2^{n-1} - 1} \left(2^n \int_{\mathbf{I}^n} C(\mathbf{u}) dC(\mathbf{u}) - 1 \right).$$

Then, we have the following corollary:

COROLLARY 3.3. *Under the hypotheses of Theorem 3.1, the measure of multivariate association $\tau_{n,C}$ is given by*

$$\tau_{n,C} = 1 - \frac{2^n}{2^{n-1} - 1} \sum_{i=1}^{n-1} \frac{(-1)^i}{i!} \int_0^1 \frac{[\phi(t)]^i \alpha_{i-1}(t)}{\phi'(t)} dt. \tag{4}$$

Remark: It is easy to check by using (4) that the expression of the well-known nonparametric measure Kendall's τ for a bivariate Archimedean copula with generator ϕ is given by $\tau = 1 + 4 \int_0^1 \phi(t)/\phi'(t) dt$ (see [2] and [3]).

4 Conclusion.

We finish this contribution with an open problem. It is well-known that for any bivariate copula C, the property of $C(u, u) < u$ for all u in (0,1) and associativity characterize the fact that C is Archimedean (see [6]). Does this result remain true for any Archimedean n-copula (or n-quasi-copula)?.

References

[1] Alsina, C.; Nelsen, R. B.; and Schweizer, B. (1993), "On the characterization of a class of binary operations on distribution functions", *Statist. Probab. Lett.*, **17**, 85-89.

[2] Genest, C.; and MacKay, R. J., (1986a), "Copules archimédiennes et familles de lois bidimensionelles dont les marges sont données", *Canad. J. Statist.*, **14**, 145-159.

[3] Genest, C.; and MacKay, R. J., (1986b), "The joy of copulas: Bivariate distributions with uniform marginals", *Amer. Statist.*, **40**, 280-285.

[4] Genest, C.; Quesada Molina, J. J.; Rodríguez Lallena, J. A.; and Sempi, C., (1999), "A Characterization of Quasi-copulas", *J. Multivariate Anal.* **69**, 193-205.

[5] Genest, C.; Quesada Molina, J. J.; Rodríguez Lallena, J. A.; and Sempi, C., (2000), "Multivariate Quasi-copulas", to appear.

[6] Ling, C.-H., (1965), "Representation of associative functions", *Publ. Math. Debrecen* **12**, 189-212.

[7] Nelsen, R. B., (1996), "Nonparametric Measures of Multivariate Association", in *Distributions with Fixed Marginals and Related Topics* (L. Rüschendorf, B. Schweizer, M. D. Taylor, Eds.), IMS Lecture Notes-Monograph Series Number 28, pp. 223-232, Hayward, CA.

[8] Nelsen, R. B., (1999), *An Introduction to Copulas*. Springer-Verlag. New York.

[9] Nelsen, R. B.; Quesada Molina, J. J.; Rodríguez Lallena, J. A.; and Úbeda Flores, M., "Best-possible bounds on Sets of Bivariate Distribution Functions", to appear.

[10] Nelsen, R. B.; Quesada Molina, J. J.; Schweizer, B.; and Sempi, C., (1996), "Derivability of some operations on distribution functions", in *Distributions with Fixed Marginals and Related Topics* (L. Rüschendorf, B. Schweizer, M. D. Taylor, Eds.), IMS Lecture Notes-Monograph Series Number 28, pp. 233-243, Hayward, CA.

[11] Schweizer, B.; and Sklar, A., (1983), *Probabilistic Metric Spaces*. Elsevier. New York.

[12] Sklar, A., (1959), "Fonctions de répartition à n dimensions et leurs marges", *Publ. Inst. Statist. Univ. Paris* **8**, 229-231.