## Separation Axioms - homework solutions

**Problem #1.** Let $(X, \mathcal{T})$ be a topological space. Let $Y = X \times X$ be the Cartesian product. We will say $\mathcal{V} \subseteq Y$ is *open* if it can be written as a union of sets of the form $\mathcal{U}_1 \times \mathcal{U}_2$ where $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{T}$. Show that this definition actually defines a topology on $Y$.

**Solution.** The empty set $\emptyset = \emptyset \times \emptyset$ and the whole space $X \times X$ are open because they take the form of $\mathcal{U}_1 \times \mathcal{U}_2$. "Open sets" are closed under arbitrary unions because unions of unions of "rectangles" are still unions of "rectangles". Finally, let us consider a finite intersection of open sets in $X \times X$. As the union distributes over the intersection it is enough to show that the intersection of finitely many basic sets is open. This easily follows from

$$\bigcap_{i=1}^{n} (\mathcal{U}_{i1} \times \mathcal{U}_{i2}) = \Big(\bigcap_{i=1}^{n} \mathcal{U}_{i1}\Big) \times \Big(\bigcap_{i=1}^{n} \mathcal{U}_{i2}\Big).$$

**Problem #2.** Show that $\mathbb{R}^2$ equipped with the topology as in problem #1 is *homeomorphic* to the regular Euclidean plane.

**Solution.** We will show that $Id : (\mathbb{R}^2, \mathcal{T}') \to (\mathbb{R}^2, \mathcal{T}_{Eucl})$ is a homeomorphism, where $\mathcal{T}'$ is the topology from Problem #1. This amounts to showing that a set is open with respect to Euclidean topology if and only if it is open with respect to the product topology $\mathcal{T}'$.
Let $\mathcal{U}$ be an open set with respect to Euclidean topology, and let $(a, b) \in \mathcal{U}$. Then there exists an open ball of radius $r$ with

$$(a, b) \subseteq B_r(a, b) \subseteq \mathcal{U}.$$

The rectangle $R_{ab} := (a - \frac{r}{2}, a + \frac{r}{2}) \times (b - \frac{r}{2}, b + \frac{r}{2})$ is completely contained within $B_r(a, b)$ since all of its elements $(x, y)$ satisfy

$$(x - a)^2 + (y - b)^2 < \Big(\frac{r}{2}\Big)^2 + \Big(\frac{r}{2}\Big)^2 < r^2.$$

It now follows that $\mathcal{U} = \bigcup_{(a,b)\in\mathcal{U}} R_{ab}$ is open in topology $\mathcal{T}'$.
Conversely, let $\mathcal{U}$ be open with respect to $\mathcal{T}'$ topology. Let $(a, b) \in \mathcal{U}$. Then there exists a rectangle $(\alpha, \beta) \times (\gamma, \delta) \subseteq \mathcal{U}$ containing $(a, b)$. Consider

$$r = \min\{a - \alpha, \beta - a, b - \gamma, \delta - b\};$$

we have $(a-r, a+r) \times (b-r, b+r) \subseteq (\alpha, \beta) \times (\gamma, \delta) \subseteq \mathcal{U}$ and as a consequence:

$$B_r(a, b) \subseteq (a - r, a + r) \times (b - r, b + r) \subseteq \mathcal{U}.$$

Since we can circumscribe an open ball around each of the elements of $\mathcal{U}$, we see that $\mathcal{U}$ is open with respect to Euclidean topology.

**Problem #3.** Show that if $(X, \mathcal{T})$ is $T_1$, so is the topological space $X \times X$ defined in problem #1. Then repeat the problem for the separation axiom $T_2$.

**Solution.** Let $(a,b), (c,d) \in X \times X$ be two distinct points in $X \times X$. Then $a \neq c$ or $b \neq d$. Without loss of generality we will assume that $a \neq c$. Since $X$ is $T_1$ (resp. $T_2$) there exist open sets $\mathcal{U}_a \ni a$ and $\mathcal{U}_c \ni c$ such that $\mathcal{U}_a \not\ni c$ and $\mathcal{U}_c \not\ni a$ (resp. $\mathcal{U}_a \cap \mathcal{U}_c = \emptyset$). The sets $\mathcal{U}_a \times X$ and $\mathcal{U}_c \times X$ are open in $X \times X$. It is immediate that $(a,b) \in \mathcal{U}_a \times X$ and that $(c,d) \in \mathcal{U}_c \times X$.

a) In the case when $X$ is $T_1$ we also have $(c,d) \notin \mathcal{U}_a \times X$ and $(a,b) \notin \mathcal{U}_c \times X$, which shows that $X \times X$ is $T_1$.

b) In the case when $X$ is $T_2$ we have

$$\big(\mathcal{U}_a \times X\big) \cap \big(\mathcal{U}_b \times X\big) = \big(\mathcal{U}_a \cap \mathcal{U}_b\big) \times X = \emptyset \times X = \emptyset,$$

which shows that $X \times X$ is $T_2$.

**Problem #4.** Show that a topological space $(X, \mathcal{T})$ is $T_2$ if and only if the "diagonal"

$$\Delta = \{(x,x) \bigm| x \in X\} \subseteq X \times X$$

is closed in $X \times X$. Use the topology of problem #1.

**Solution.** First assume $X$ is $T_2$. I will show that the complement of the diagonal in $X \times X$ is open in the sense of Problem #1. Let $(a,b) \notin \Delta$; this means that $a \neq b$. Since $X$ is $T_2$ we have disjoint open sets $\mathcal{U}_a \ni a$ and $\mathcal{U}_b \ni b$. Consider the open rectangle $\mathcal{U}_a \times \mathcal{U}_b$. Since

$$\big(\mathcal{U}_a \times \mathcal{U}_b\big) \cap \Delta = \big\{(c,c) \bigm| c \in \mathcal{U}_a \cap \mathcal{U}_b\big\} = \big(\mathcal{U}_a \cap \mathcal{U}_b\big) \times \big(\mathcal{U}_a \cap \mathcal{U}_b\big) = \emptyset,$$

we see that $(a,b) \in \mathcal{U}_a \times \mathcal{U}_b \subseteq \big(X \times X\big) \smallsetminus \Delta$. As a consequence

$$\big(X \times X\big) \smallsetminus \Delta = \bigcup_{(a,b) \notin \Delta} \mathcal{U}_a \times \mathcal{U}_b,$$

and $\Delta$ is closed.

Conversely, assume $\Delta \subseteq X \times X$ is closed. This means that for each $(a,b) \notin \Delta$ there is an open rectangle $\mathcal{U}_a \times \mathcal{U}_b$ such that

$$(a,b) \in \mathcal{U}_a \times \mathcal{U}_b \subseteq \big(X \times X\big) \smallsetminus \Delta.$$

The sets $\mathcal{U}_a$ and $\mathcal{U}_b$ are open neighborhoods of $a$ and $b$ respectively. If $x$ were an element of $\mathcal{U}_a \cap \mathcal{U}_b$ then $(x,x) \in \big(\mathcal{U}_a \times \mathcal{U}_b\big) \cap \Delta = \emptyset$. Thus $\mathcal{U}_a \cap \mathcal{U}_b = \emptyset$ and $X$ is Hausdorff.

**Problem #5.** Let $f, g : X \to Y$ be two continuous functions between two topological spaces. Assume $Y$ is $T_2$.

(1) Show the set
$$\{x \in X \mid f(x) = g(x)\}$$
is closed in $X$.

(2) Show that if $f$ and $g$ agree on a dense subset of $X$, then they agree on the entire set $X$.

**Solution.** Consider the function

$$F : X \to Y \times Y, \quad \text{given by} \quad F(x) = \big(f(x), g(x)\big).$$

This function is continuous. Indeed, let $\mathcal{U} = \bigcup_\alpha \big(\mathcal{V}_\alpha \times \mathcal{W}_\alpha\big) \subseteq Y \times Y$ be open. The pre-image of $\mathcal{U}$ under $F$,

$$F^{-1}(\mathcal{U}) = \bigcup_\alpha F^{-1}(\mathcal{V}_\alpha \times \mathcal{W}_\alpha),$$

is open in $X$ since $F^{-1}(\mathcal{V}_\alpha \times \mathcal{W}_\alpha\big) = f^{-1}(\mathcal{V}_\alpha) \cap g^{-1}(\mathcal{W}_\alpha)$ is open as an intersection of two pre-images of open sets under continuous maps.

(1) By assumption $Y$ is Hausdorff, and by the previous problem the diagonal $\Delta \subseteq Y \times Y$ is closed. Since $F$ is continuous the pre-image of the diagonal must be closed. So,

$$F^{-1}(\Delta) = \big\{x \big| \ \big(f(x), g(x)\big) \in \Delta\big\} = \{x \in X \mid f(x) = g(x)\}$$

must be closed in $X$.

(2) Let $D \subseteq X$ be a dense set on which the functions $f$ and $g$ agree. Then $D$ is contained in the set $\{x \in X \mid f(x) = g(x)\}$, which we now know is closed. Since $Cls(D) = X$ is *the smallest* closed subset which contains $D$ we have

$$X \subseteq \{x \in X \mid f(x) = g(x)\}, \quad \text{i.e.} \quad f(x) = g(x) \quad \text{for all } x \in X.$$