CAUCHY SEQUENCES

Here is a notion which is crucial to extending the notion of completeness to metric spaces, which we will study later in the semester.

Definition. *A sequence of real numbers* (s_n) *is said to be a Cauchy sequence if*

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n, m \in \mathbb{N}) \quad n, m \geq N \Longrightarrow \big|s_n - s_m\big| < \varepsilon.$$

This definition resembles the definition of the limit, but is very different in that the value of the limit does not appear in this definition. Rather than saying that the terms of the sequence "accumulate" around a certain value, this definition refers to terms of our sequence "accumulating" around each other. In other words, a sequence is Cauchy if its terms are as close to each other as desired (ε) assuming we are far enough down the sequence (N).

Example. Show that the sequence $s_n = \frac{\sin(n^2)}{n}$ is Cauchy.

Comments leading to the solution. For given $\varepsilon > 0$ we need to find N such that all $n, m \geq N$ satisfy

$$\Big|\frac{\sin(n^2)}{n} - \frac{\sin(m^2)}{m}\Big| < \varepsilon.$$

In fact, due to the triangle inequality and the properties of the sin-function, it suffices to find N such that for all $n, m \geq N$ we have

$$\frac{1}{n} + \frac{1}{m} < \varepsilon.$$

Since $\frac{1}{n} + \frac{1}{m} \leq \frac{2}{N}$ we can choose $N > \frac{2}{\varepsilon}$.

Solution. Let $\varepsilon > 0$ and consider $N \in \mathbb{N}$ with $N > \frac{2}{\varepsilon}$. If $n, m \geq N$ then

$$\Big|\frac{\sin(n^2)}{n} - \frac{\sin(m^2)}{m}\Big| \leq \Big|\frac{\sin(n^2)}{n}\Big| + \Big| - \frac{\sin(m^2)}{m}\Big| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that our sequence is Cauchy.

Here is a **very important** theorem. Its proof is absolutely brilliant.

Theorem. *A sequence is Cauchy if and only if it is convergent.*

Proof. We first prove that a convergent sequence (s_n) is also Cauchy. Let $\lim\limits_{n\to\infty} s_n = s$ and let $\varepsilon > 0$. By the definition of convergence we know that there is some $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|s_n - s| < \tfrac{\varepsilon}{2}.$$

Then in particular,

$$|s_n - s_m| = |(s_n - s) - (s_m - s)| \leq |s_n - s| + |s_m - s| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

This shows that the sequence is Cauchy.

We now show that **if** a sequence is Cauchy then it is also convergent. Consider the set

$$S = \{x \in \mathbb{R} \mid s_n < x \ \text{ for only finitely many } n\}.$$

We shall show that $L = \sup S$ exists and that $\lim_{n\to\infty} s_n = L$. Let $\varepsilon > 0$. Since (s_n) is Cauchy we know there is some $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|s_m - s_n| < \frac{\varepsilon}{2}$. This means that all $n \geq N$ satisfy

$$s_N - \tfrac{\varepsilon}{2} < s_n < s_N + \tfrac{\varepsilon}{2}.$$

Consequently, there are only finitely many n with $s_n < s_N - \frac{\varepsilon}{2}$ and there are infinitely many n with $s_n < s_N + \frac{\varepsilon}{2}$. In other words,

$$s_N - \tfrac{\varepsilon}{2} \in S, \qquad s_N + \tfrac{\varepsilon}{2} \notin S.$$

We now may conclude that $s_N + \frac{\varepsilon}{2}$ is an upper bound for S. Indeed, if there were some $x \in S$ with $s_N + \frac{\varepsilon}{2}$ then we would also have

$$s_n < s_N + \tfrac{\varepsilon}{2} < x \quad \text{for all} \quad n \geq N.$$

In other words, there would be infinitely many n with $s_n < x$, contrary to $x \in S$. Hence, S is bounded from above by $s_N + \frac{\varepsilon}{2}$. By **Completeness Axiom** we know that $L = \sup S$ exists. Since $s_N - \frac{\varepsilon}{2} \in S$ and since $s_N + \frac{\varepsilon}{2}$ is an upper bound for S we must have

$$s_N - \tfrac{\varepsilon}{2} \leq L \leq s_N + \tfrac{\varepsilon}{2}.$$

To show that $\lim_{n\to\infty} s_n = L$ it suffices to argue that $|s_n - L| < \varepsilon$ for all $n \geq N$. This inequality follows from

$$|s_n - L| = |(s_n - s_N) + (s_N - l)| \leq |s_n - s_N| + |s_N - L| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

This completes the proof of our theorem. □

You must have noticed that Completeness Axiom plays a crucial role in the above proof. In fact, the theorem would not hold if there was no Completeness Axiom. One can easily find Cauchy sequences of rational numbers which do not converge in $\mathbb{Q}$[1]. For example, consider the sequence informally described by

$$s_1 = 1.4, \quad s_2 = 1.41, \quad s_3 = 1.414, \quad s_4 = 1.4142, \ldots.$$

Formally, the terms of this sequence are chosen so that

$$s_{n+1} - s_n < 10^{-n} \ \text{ along with } \ {s_n}^2 < 2 \ \text{ and } \ (s_n + 10^{-n})^2 \geq 2.$$

Loosely speaking, think of s_n as being a rational approximation of $\sqrt{2}$, accurate to $10^{-(n-1)}$. One can show that this sequence is Cauchy: for $\varepsilon > 0$ chose $N \in \mathbb{N}$ such that $10^{-N} < \varepsilon$ and note that all $m, n \geq N$ satisfy $\left|s_m - s_n\right| < 10^{-N} < \varepsilon$. By construction our sequence is such that $s_n \to \sqrt{2}$ as $n \to \infty$. In other words, this sequence is Cauchy in $\mathbb{Q}$ but not convergent in $\mathbb{Q}$.

Why is this theorem **SO** important? The statement that Cauchy sequences converge is possible only because the set of real numbers $\mathbb{R}$ satisfies the Completeness Axiom. This axiom though relies on the notions of supremum and $\leq$. There are plenty of "spaces" of interest where we need to do analysis / advanced calculus, but we have **no** notion of $\leq$ (much less supremum)! The simplest examples of such "spaces" are the plane $\mathbb{R}^2$ and the space $\mathbb{R}^3$. Instead of imposing Completeness Axiom in terms of supremum, in "spaces" such as $\mathbb{R}^2$ and $\mathbb{R}^3$ we impose that all Cauchy sequences converge. In other words, the theorem we proved above is a gateway to the notion of completeness in more abstract set-ups.

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[1] The set $\mathbb{Q}$ does not satisfy the Completeness Axiom.